In this lecture we will look at solving a particular type of ODE, which can be written in the form

\[ \frac{dy}{dx} = f(x, y) \]  

where \( f(x, y) \) is a particular type of function, namely one which is homogeneous of degree zero, which we will define below. More general types of homogeneous functions are interesting for algebraic reasons, so we will define what we mean here by “homogeneous” in Sections 1 and 2. Next, in Section 3 we will show how an equation (1), with \( f \) homogeneous of degree zero, can be separated after a clever change of variable, and therefore solved. Section 4 summarizes the actual method, and the final section shows two examples. As usual the method, while fairly straightforward, requires some care algebraically. We will also need to be alert to the possibility of near-trivial solutions.

In the greater scheme of ODE methods, this one is not especially difficult. It really contains one clever trick to learn, and otherwise just requires the usual care and somewhat involved calculus.

1 Definition of Homogeneous of Degree \( \alpha \)

A function \( f(x, y) \) of two variables is called homogeneous of degree \( \alpha \) if

\[ f(tx, ty) = t^\alpha f(x, y). \]  

One can usually spot such functions and their degrees with some practice. Consider the following examples:

(a) \( f(x, y) = x^2 + 5xy + y^2 \) is homogeneous of degree 2, since

\[
\begin{align*}
f(tx, ty) &= (tx)^2 + 5(tx)(ty) + (ty)^2 \\
&= t^2x^2 + t^2(5xy) + t^2y^2 \\
&= t^2(x^2 + 5xy + y^2) \\
&= t^2f(x, y).
\end{align*}
\]

(b) \( f(x, y) = x^4y^2 + x^3y^3 + \frac{y^7}{x} \) is homogeneous of degree 6, since

\[
\begin{align*}
f(tx, ty) &= (tx)^4(ty)^2 + (tx)^3(ty)^3 + \frac{(ty)^7}{tx} \\
&= t^6x^4y^2 + t^6x^3y^3 + \frac{t^7y^7}{tx} \\
&= t^6 \left( x^4y^2 + x^3y^3 + \frac{y^7}{x} \right) \\
&= t^6f(x, y).
\end{align*}
\]

\footnote{It should be pointed out that the term homogeneous has many definitions in mathematics. Unlike our previous use of the word, referring to a linear ODE of the form \( L[y] = 0 \), here it corresponds to the idea of “uniform structure.”}
Note that homogeneity is a question of total powers of $x$ and $y$. A term $x^n y^m$ would be homogeneous of degree $n + m$. It is very much like treating $x$ and $y$ as the same variable and seeing what power of the variable we are left with. A term like $x^n y^{-m}$ would be homogeneous of degree $n - m$.

We can only add and subtract terms with the same homogeneity if we would like to preserve that homogeneity. For instance $x^2 y^4 + x^3 y$ would be a sum of degree 6 and degree 4, and would collectively not be homogeneous of any degree. Similarly $\frac{x y + 3x^2 + 9y^2}{xy}$ is not homogeneous either, since the denominator is homogeneous of degree two, but the numerator is inhomogeneous. (To be convinced, try to factor all powers of $t$ from $f(tx, ty)$ and see how it cannot be of the form $t^a f(x, y)$: it is impossible!)

Some texts only require the definition (2) to hold for $t > 0$, which allows us to call $f(x, y) = \sqrt{x^2 + y^2}$ homogeneous of degree 1, since

$$f(tx, ty) = \sqrt{(tx)^2 + (ty)^2} = \sqrt{t^2(x^2 + y^2)} = |t| \sqrt{x^2 + y^2} = t^1 f(x, y), \quad t > 0.$$ 

This is somewhat intuitive, because the $x^2 + y^2$ is homogeneous of degree 2, while the radical represents a one-half power. In algebra we learn that $(a^n)^m = a^{mn}$, at least for integers $m, n$, so this kind of multiplication of degrees mimics that rule.

If one is interested in units (also known as dimensions), it should be noted that a homogeneous function will return consistent, uniform units when $x$ and $y$ are given the same units. This is clarified in the next example.

**Example 1** Suppose $x$ and $y$ are both given in feet, i.e., ft. Then we can say the following about our previous examples:

(a) $f(x, y) = x^2 + 5xy + y^2$ will return units of ft$^2$.

(b) $f(x, y) = x^4 y^2 + x^3 y^3 + \frac{y^7}{x}$ will return units of ft$^6$.

(c) $f(x, y) = \frac{x}{x^2 + y^2} + \frac{1}{y} \sin(x/y)$ will return units of ft$^{-1}$.

Recall these functions were homogeneous of degrees 2, 6 and $-1$, respectively.

Consider any function $f(x, y)$ which is not homogeneous of any degree. Then assigning the same units to $x$ and $y$ will not yield consistent units in the output. For example, $f(x, y) = x^2 + x + 1/y$ would have an output attempting to add units of ft$^2$, ft and ft$^{-1}$, which is physically impossible. This suggests that homogeneous functions may be quite important in the physical sciences, and elsewhere that units are important.\(^2\)

\(^{2}\)One might think from this consideration of units that only homogeneous functions would arise in the physical sciences. This is not true, because many functions in the physical sciences input variables which have different units, and contain physical constants which themselves contain the necessary units to reconcile the variables and give the appropriate units to the output. For example, one could look at the Ideal Gas Law, $PV = nkT$, in the context of functions, with $P = P(V, n, T) = nkT/V$. Here the constant $k$ contains the proper units to give the output in units of pressure, though $V$ is in volume, $n$ in number of particles and $T$ in absolute temperature. That said, using units to understand homogeneity is still useful, even if it is not the end of the story.
2 Functions Homogeneous of Degree Zero

For the ODE technique we are interested in for this lecture we consider functions \( f(x, y) \) which are **homogeneous of degree zero**, formally meaning that \( f(tx, ty) = t^0 f(x, y) \), but we will assume the following for all \( t \neq 0 \) (even though \( t^0 \) really only makes sense if \( t > 0 \)), we have

\[
f(tx, ty) = f(x, y).
\]

These are easy to spot as well. Functions such as \( f(x, y) = \frac{5x}{y} \), \( f(x, y) = x^2 + y^2 \), and the like are such functions, for if we replace \((x, y)\) with \((tx, ty)\), we do not change the function values. Another such function would be \( f(x, y) = x^3y + 6x^2y^2 - x^2y^2 + \tan(x/y) + 5 \).

Several pages could be written on the algebraic and geometric considerations of functions which are homogeneous of degree zero. For instance, it is again interesting to consider these functions in the context of units. Note that if \( x \) and \( y \) were given the same units, such as feet (ft), then a function \( f(x, y) \) which is homogeneous of degree zero will have a unitless (or dimensionless) output \((\text{ft}^0)\). This is interesting in its own right, since it means that it does not matter what are the units of \( x \) and \( y \), as long as they are the same.

One other interesting (and later crucial) algebraic result regarding functions homogeneous of degree zero is the following:

**Theorem 1** If \( f(x, y) \) is homogeneous of degree zero, then we can write \( f(x, y) \) as a function of the ratio \( y/x \), i.e., there is a function \( g \) of a single variable so that

\[
f(x, y) = g\left(\frac{y}{x}\right).
\]

In other words, if you know \( y/x \) (i.e., know which line through the origin \((x, y)\) lies on), then you know \( f(x, y) \). For the proof, we “simply” write

\[
f(x, y) = f(x \cdot 1, x \cdot \frac{y}{x}) = x^0 f\left(1, \frac{y}{x}\right) = f\left(1, \frac{y}{x}\right).
\]

Here the part of \( t \) in the definition of homogeneity is played by \( x \). Also note that the above is indeed only dependent upon \( y/x \) as claimed (since “1 is always 1”). Indeed, if we define a function \( g \) by

\[
g(s) = f(1, s),
\]

we have

\[
f(x, y) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right).
\]

This completes the proof.\(^3\)

A proof like the above may require some pondering before it loses its sleight-of-hand appearance and finally seems believable, but it is correct and an examination will find no errors.

---

\(^3\)A function in \( x \) and \( y \), homogeneous of degree zero, and therefore a function of \( y/x \), has some interesting geometric properties.

(a) If we write the function instead in polar coordinates, say \( f(x, y) = \phi(r, \theta) \), then it is constant along lines \( \theta = \theta_0 \), i.e., \( y/x = \tan \theta_0 \), and so \( f(x, y) = \phi(r, \theta) = \Phi(\theta) \), and not depending upon the radial coordinate. So \( f(x, y) \) is just a function of the polar angle \( \theta \).

(b) Therefore, unless that function is constant (which is one example of a degree-zero homogeneous function), it is necessarily discontinuous at the origin \((x, y) = (0, 0)\), i.e., \( r = 0 \).

(c) If \( x \) and \( y \) are given the same dimensions, such as feet, then \( f(x, y) \) will be dimensionless (feet cancel). This is worth noting as we work through examples later.
3 Theory and Separation

In this short section we show how the equation
\[ \frac{dy}{dx} = f(x, y), \quad \text{where } f \text{ is homogeneous of degree zero}, \] (5)
can be separated, after a “change of variable.” The final method will be contained in this derivation, and summarized at the start of Section 4. It is basically a substitution-type argument. We give its derivation here.

With the help of Theorem 1, we can rewrite our ODE (5) as follows:
\[ \frac{dy}{dx} = g(y/x). \] (6)

The next, and key step is to make a substitution:
\[ u = \frac{y}{x}. \] (7)

Immediately we can see (6) becomes \( \frac{dy}{dx} = g(u) \), but this seemingly has three variables instead of two. This is not wholly true since \( u \) depends on both \( x \) and \( y \) algebraically. Nonetheless we need our ODE to have only two variables, and the more useful form of \( u = y/x \) for that purpose is the following, basically replacing \( y \) using
\[ y = ux. \] (8)

Now we differentiate (8) with respect to \( x \), i.e., apply \( \frac{d}{dx} \) to both sides. Using the product rule we get
\[ \frac{dy}{dx} = u + x \frac{du}{dx}, \] (9)
With this our ODE (6) becomes
\[ u + x \frac{du}{dx} = g(u). \] (10)

This is fairly easily separated, the first steps being
\[ (10) \implies x \frac{du}{dx} = g(u) - u, \] (11)
\[ \implies \frac{du}{dx} = \frac{g(u) - u}{x}. \] (12)

At this point we have to be careful to include those cases of constant \( u \), i.e., \( u = k \) which cause the RHS of (12) to be zero (and recall that \( u = k \) would make LHS of (12) zero as well), and are thus more trivial solutions to the new ODE. Putting those aside for the moment, we next separate variables and integrate to get a solution in \( u \) and \( x \):
\[ \int \frac{du}{g(u) - u} = \int \frac{dx}{x} \] (13)
\[ \implies G(u) = \ln |x| + C, \] (14)

where \( G(u) \) is some antiderivative of \( 1/(g(u) - u) \) in \( u \), i.e., \( G'(u) = 1/(g(u) - u) \). The last thing to do is replace \( u \) using \( u = y/x \), giving us
\[ G \left( \frac{y}{x} \right) = \ln |x| + C, \] (15)
as well as the \( u = k \) solutions to (12), i.e., \( y/x = k \), i.e.,
\[ y = kx \] (16)
solutions. Of course there may be some desirable algebra to consider in order to simplify the presentation of the general solution, such as applying the exponential function \( \exp \) to both sides of (15) to rid ourselves of the natural log in the RHS.\(^4\)

\(^4\)One really should check any solutions in the original ODE (5) to be sure, through all the divisions etc., that such
4 Method and Examples

A method is embedded in the derivation of the previous section. Assuming we have already written the equation into the form (1), i.e.,
\[ \frac{dy}{dx} = f(x, y) \]
we then do the following:

1. Verify that \( f(x, y) \) is homogeneous of degree zero. (If not, try to find another method.) In other words, check
\[ f(tx, ty) = f(x, y). \]

2. Substitute \( y = ux \) in LHS and RHS of the ODE.
   (a) LHS becomes \( u + x \frac{du}{dx} \).
   (b) RHS will become a function of \( u \) alone. (There are two techniques for accomplishing this.)

3. Solve algebraically for \( \frac{du}{dx} \), noting the possible “constant \( u \)” solutions.

4. Separate and solve the new form of the ODE.

5. Replace \( u = y/x \) in all solutions. DONE (except for possible simplifying, and checking that solutions make sense in original ODE, which is rarely not the case with these particular ODE's).

Example 2 Solve the (non-separable, nonexact, nonlinear) ODE
\[ \frac{dy}{dx} = \frac{x + y}{x - y}, \quad (17) \]

First we can see \( f(x) = (x + y)/(x - y) \) is homogeneous of degree zero, though we could check:
\[ f(tx, ty) = \frac{tx + ty}{tx - ty} = \frac{t(x + y)}{t(x - y)} = \frac{x + y}{x - y} = f(x, y). \]

That being verified (or just noted), we now substitute \( y = ux \) \( \Rightarrow \) \( \frac{dy}{dx} = u + x \frac{du}{dx} \). Our ODE (17) becomes
\[ \frac{d(ux)}{dx} = \frac{x + ux}{x - ux} \]

\[ \Rightarrow u + x \frac{du}{dx} = \frac{x(1 + u)}{x(1 - u)} \]

\[ \Rightarrow u + \frac{du}{dx} = \frac{1 + u}{1 - u}. \]

Solving for \( x \frac{du}{dx} \) (and then combining RHS into one fraction) gives
\[ \frac{du}{dx} = \frac{1 + u}{1 - u} - u = \frac{1 + u - u(1 - u)}{1 - u} = \frac{u^2 + 1}{1 - u}. \]

Note the form on the right is “\( g(u) - u \)” as in our theory. Now we solve the separable ODE
\[ x \frac{du}{dx} = \frac{u^2 + 1}{1 - u}. \quad (18) \]

\( y \)'s do not cause RHS of the original ODE to be undefined, for instance. There are also cases where the constant \( u \) solutions, i.e., \( y = tx \) solutions reappear when we simplify (15), and possibly cases where these give \( dy/dx \) to be undefined in the original ODE. As a general rule the answer has to be consistent with the original ODE at some points. Indeed, when we perform algebraic steps in between calculus steps, for instance, there can be the danger that a problem in the original ODE is lost in those calculations, or a more trivial solution is lost in the same calculations.
There are no "constant u" solutions here, since \(u^2 + 1 \neq 0\). After separation we get

\[
\frac{1 - u}{u^2 + 1} \, du = \frac{dx}{x}
\]

\[
\Rightarrow \quad \int \left( \frac{1}{u^2 + 1} - \frac{u}{u^2 + 1} \right) \, du = \int \frac{dx}{x}
\]

\[
\Rightarrow \quad \tan^{-1} u - \frac{1}{2} \ln(u^2 + 1) = \ln|x| + C_1.
\]

Substituting \(u = y/x\) then gives us a passable solution:

\[
\tan^{-1} \frac{y}{x} - \frac{1}{2} \ln \left( \left( \frac{y}{x} \right)^2 + 1 \right) = \ln|x| + C_1.
\]

One algebraic step which could now be taken would be to combine the fraction inside the log and rewrite the log on the RHS as \(\ln(x^2)^{1/2} = \frac{1}{2} \ln x^2\):

\[
\tan^{-1} \frac{y}{x} - \frac{1}{2} \ln \frac{y^2 + x^2}{x^2} = \frac{1}{2} \ln x^2 + C_1.
\]

Expanding the log on our current LHS gives

\[
\tan^{-1} \frac{y}{x} - \frac{1}{2} \ln(y^2 + x^2) + \frac{1}{2} \ln x^2 = \frac{1}{2} \ln x^2 + C_1.
\]

We see that the \(\ln x^2\) terms cancel, and we get

\[
\tan^{-1} \frac{y}{x} - \frac{1}{2} \ln(y^2 + x^2) = C.
\]

Before we finish this example, we note an alternative method for writing the RHS of the ODE (17) in terms of \(u = y/x\), by dividing the numerator and denominator by \(x\) (or \(x\) to the power of the homogeneity of each):

\[
\frac{x + y}{x - y} \cdot \frac{1}{x} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}} = \frac{1 + u}{1 - u}.
\]

Technically the above example was solved at (19), but a little algebraic ingenuity helped simplify the solution. However the solution, even as finally presented in (22) is still given by a family of level curves of a function, i.e., \(F(x,y) = C\) where \(F(x,y) = \tan^{-1} (y/x) - \frac{1}{4} \ln(y^2 + x^2)\). As we’ve seen before, this is very typical with ODE’s; we are lucky when we can solve for \(y = y(x)\)!

**Example 3** Solve the ODE

\[
\frac{dy}{dx} = \frac{2y^2 - 22xy}{xy + 3x^2}.
\]

First we check that \(f(x,y) = (2y^2 - 22xy)/(xy + 3x^2)\) is homogeneous of degree zero. It is, as the calculation below confirms (again note it is reasonable to check visually; see Item (c) in Footnote 3).

\[
f(tx,ty) = \frac{2(ty)^2 - 22t^2xy}{t^2xy + 3(tx)^2} = \frac{t^2(2y^2 - 22xy)}{t^2(xy + 3x^2)} = \frac{2y^2 - 22xy}{xy + 3x^2} = f(x,y).
\]
Next we substitute \( y = xu \), and so \( y = ux \) \( \Rightarrow \) \( \frac{dy}{dx} = u + x \frac{du}{dx} \), and (23) becomes

\[
\begin{align*}
    u + x \frac{du}{dx} &= \frac{2(ux)^2 - 22x(uu)}{x(uuu) + 3x^2} \\
    \Rightarrow u + x \frac{du}{dx} &= \frac{x^2(2u^2 - 22u)}{x^2(u + 3)} \\
    \Rightarrow u + x \frac{du}{dx} &= \frac{2u^2 - 22u}{u + 3} \\
    \Rightarrow x \frac{du}{dx} &= \frac{2u^2 - 22u - u^2 - 3u}{u + 3} \\
    \Rightarrow x \frac{du}{dx} &= \frac{u^2 - 25u}{u + 3} \\
    \Rightarrow \frac{du}{dx} &= \frac{1}{x} \cdot \frac{u^2 - 25u}{u + 3}.
\end{align*}
\]

(Note the alternative method for computing the RHS of (23) as a function of \( u \), here namely multiplying numerator and denominator there by \( 1/x^2 \).) From here we see that any constant solution of \( u(u - 25) = 0 \) gives zero on the right, so \( u = 0 \), \( 25 \) are both solutions to this new ODE. Now we look for others by separation:

\[
\begin{align*}
    u + 3 &= \frac{u + 3}{u^2 - 25u} \cdot \frac{du}{dx} = \frac{1}{x} \cdot \frac{u + 3}{u(u - 25)} \\
    \Rightarrow \int u + 3 \frac{du}{u(u - 25)} &= \int \frac{1}{x} \cdot \frac{u + 3}{u(u - 25)} dx.
\end{align*}
\]

The integral on the left requires partial fractions:

\[
\frac{u + 3}{u(u - 25)} = \frac{A}{u} + \frac{B}{u - 25} \\
\Rightarrow u + 3 = A(u - 25) + Bu
\]

\[
\begin{align*}
    u = 0 : & \quad 3 = -25A \quad \Rightarrow \quad A = -3/25 \\
    u = 25: & \quad 28 = 25B \quad \Rightarrow \quad B = 28/25
\end{align*}
\]

This all gives us

\[
\begin{align*}
    \int \left( \frac{-3/25}{u} + \frac{28/25}{u - 25} \right) du &= \int \frac{1}{x} \cdot \frac{u + 3}{u(u - 25)} dx.
\end{align*}
\]

The solution in \( u, x \) is then

\[
-\frac{3}{25} \ln |u| + \frac{28}{25} \ln |u - 25| = \ln |x| + C_1.
\]

One thing that we can do here is combine all the logarithms:

\[
\begin{align*}
    \frac{1}{25} \ln \left| \frac{(u - 25)^{28}}{u^3} \right| &= \ln |x| + C_1 \\
    \Rightarrow \ln \left| \frac{(u - 25)^{28}}{u^3} \right| &= 25 \ln |x| + 25C_1 \\
    \Rightarrow \ln \left| \frac{(u - 25)^{28}}{x^{25} u^3} \right| &= C_2
\end{align*}
\]
Taking exponentials (and combining $+/-$ cases) we get
\[
\frac{(u - 25)^28}{x^{25}u^3} = C.
\]
Replacing $u = y/x$ gives
\[
C = \left(\frac{y}{x} - 25\right)^{28} \frac{1}{x^{25}(\frac{y}{x})^3} = \frac{(y - 25x)^{28}}{x^{50}y^3}.
\]
Combining this with the constant $u$-solutions $u = 0, 25$, i.e., $y/x = 0, 25$, gives us
\[
(y - 25x)^{28} = Cx^{50}y^3,
\]
\[
y = 0, \; y = 25x.
\]
If we check the original equation (23) we see that $y = 0$ is a solution, and nothing terrible goes wrong if we “plug in” $y = 25x$ so we can be sure that is also a solution (without needing to go through all the calculations to prove it). In fact, the $y = 25x$ solution is the $C = 0$ case in (24), but we can not see the $y = 0$ solution from that form (so Farlow would call it a singular solution). The moral is that we need to find these linear solutions $y = kx$ from the equation for $du/dx$ because they may not (all) appear in the final formula from separation of variables calculations. Since $y = 25x$ is contained in (24) (with $C = 0$), we can summarize the solution (24), (25) to our ODE instead as
\[
(y - 25x)^{28} = Cx^{50}y^3,
\]
\[
y = 0.
\]

**Homework 5-A**

1. For each of the following functions, decide if it is homogeneous, and if so, of what degree.

   (a) $f(x, y) = 9x^3y + 8x^2y^2 - 6xy^3$
   (b) $f(x, y) = \tan(x^2/y^2) + 4$
   (c) $f(x, y) = \frac{x^4y}{x^2y^2 - xy^3}$
   (d) $f(x, y) = 5x^2 + 9x + y^2 - 7y$
   (e) $f(x, y) = e^{x/y}$
   (f) $f(x, y) = \sqrt{xy}$

2. Write the following as a function of $u = y/x$:
\[
f(x, y) = \frac{x^3 + y^3}{3x^2y - x^2y + 9xy^2}.
\]

3. Suppose $f(x, y)$ is homogeneous of degree $m$, and $g(x, y)$ is homogeneous of degree $n$. Prove that

   (a) $f(x, y)g(x, y)$ is homogeneous of degree $m + n$.
   (b) $f(x, y)/g(x, y)$ is homogenous of degree $m - n$.

4. Show the RHS is homogeneous of degree zero and solve the following ODE:
\[
\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}.
\]