# Chapter 1

# Mathematical Logic and Sets

In this chapter we introduce symbolic logic and set theory. These are not specific to calculus, but are shared among all branches of mathematics. There are various symbolic logic systems, and indeed mathematical logic is its own branch of mathematics, but here we look at that portion of mathematical logic which should be understood by any professional mathematician or advanced student. The set theory is a natural extension of logic, and provides further useful notation as well as some interesting insights of its own.

The importance of logic to mathematics cannot be overstated.[1] No conjecture in mathematics is considered fact until it has been logically proven, and truly valid mathematical analysis is done only within the rigors of logic. Because of this dependence, mathematicians have carefully developed and formalized logic beyond some of the murkier "common sense" we learn from childhood, and given it the precision required to explore, manipulate and communicate mathematical ideas unambiguously. Part of that development is the codification of mathematical logic into symbols. With logic symbols and their rules for use, we can analyze and rewrite complicated logic statements much like we do with algebraic statements.

Symbolic logic is a powerful tool for analysis and communication, but we will not abandon written English altogether. In fact, most of our ideas will be expressed in sentences which mix English with mathematical expressions including symbolic logic. We will strive for a pleasant style of mixed prose, but we will always keep in mind the formal logic upon which we base our arguments, and resort to the symbolic logic when the logic-in-prose is complicated or can be illuminated by a symbolic representation.

Because we will use English phrases as well as symbolic logic, it is important that we clarify exactly what we mean by the English versions of our logic statements. Part of our effort in this chapter is devoted toward that end.

The symbolic language developed here is used throughout the text. It is descriptive and precise, and learning its correct use forces clarity in thinking and presentation. It is not common for a calculus textbook to include a study of logic, since authors have more than enough to accomplish in trying to offer a respectably complete treatment of the calculus itself. However, it

[1] In fact Bertrand Russell (1872–1970)—one of the greatest mathematicians of the twentieth century—argued successfully that mathematics and logic are exactly the same discipline. Indeed, they seem to be supersets of each other, implying they are the same set. It just happens that to many a lay person, mathematics may be associated only with numbers and computations while logic deals with argument. The field of geometry belies this categorization, but there are many other vast mathematical disciplines which are not so interested in our everyday number systems. These include graph theory (useful for network design and analysis), topology (used to study surfaces, relativity), and abstract algebra (used for instance in coding theory) to name a few. Indeed both mathematics and logic can be defined as interested in abstract, coherent structural systems. Thus, to a modern mathematician, logic-versus-mathematics may be considered a "distinction without a difference."

| Symbol | Read | Example | Also Read |
|---|---|---|---|
| $\sim$ | not | $\sim P$ | $P$ is not true<br>$P$ is false |
| $\wedge$ | and | $P \wedge Q$ | $P$ and $Q$<br>both $P$ and $Q$ are true |
| $\vee$ | or | $P \vee Q$ | $P$ or $Q$<br>$P$ is true or $Q$ is true (or both) |
| $\longrightarrow$ | implies | $P \longrightarrow Q$ | if $P$ then $Q$<br>$P$ only if $Q$ |
| $\longleftrightarrow$ | if and only if | $P \longleftrightarrow Q$ | $P$ bi-implies $Q$<br>$P$ iff $Q$ |

**Table 1.1:** Some basic logic notation.

is quite common for teachers and professors to insert some of the logic notation into the class lectures because of its usefulness for presenting and explaining calculus to students. Unfortunately a casual or "on the fly" introduction to these devices can cause as many problems as it solves. In this text we will instead commit early to developing and using the symbolic logic notation so we can take advantage of its correct use.

We begin with the first section (Section 1.1) devoted to the construction of truth tables, which ultimately define our first group of logic symbols. Subsequent sections in this chapter will explore valid logical equivalences (Section 1.2), valid implications and some general argument types (Section 1.3), quantifiers (Section 1.4) and sets (Section 1.5). An optional, final section (Section 1.6) considers further symbolic logic manipulations based upon those built up in the previous sections.

## 1.1 Logic Symbols and Truth Tables

The first logic symbols we develop in the text are listed in Table 1.1. In what follows we will explain their meanings and give their English versions, while also pointing out where casual English interpretations often differ, from each other as well as their formal meanings. It is useful to learn to read the symbols above as they would usually be said out loud. For instance, $P \wedge Q$ can be read, "$P$ and $Q$," while $P \longrightarrow Q$ is usually read "$P$ implies $Q$."[2] One reads $\sim P$ as "not $P$," while more elaborate means for verbalizing, say, $\sim (P \vee Q)$ would include "it is not the case that $P$ or $Q$." In fact, if $P$ is any statement, such as "it is raining," then we can graft the words "it is the case that" and have a new statement with exactly the same meaning: "it is the case that it is raining." This allows us more flexibility to read negations in a more natural order: $\sim P$ becomes "it is not the case that it is raining."

Now we look again at the symbols in Table 1.1. The symbol $\sim$ is called a *unary logic operation* because it operates on one (albeit possibly compound) statement, say $P$. The symbols $\vee, \wedge, \longrightarrow, \longleftrightarrow$ are called *connectives* or *binary logic operations*, connecting two statements, such as $P$, $Q$. Both types will be developed in detail in this chapter.

[2] Occasionally some of these are verbalized using what amounts to their typographical descriptions, so for instance $P \wedge Q$ becomes "$P$ wedge $Q$," while $P \vee Q$ becomes "$P$ vee $Q$."

| $P$ |
|---|
| T |
| F |

| $P$ | $Q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

| $P$ | $Q$ | $R$ |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

**Figure 1.1:** Lexicographical ordering of the possibilities for one, two, or three independent statements' truth values. The extra horizontal line in the third table is for ease of reading only.

### 1.1.1 Lexicographical Listings of Possible Truth Values

In the next subsection we develop the logic operators listed previously in Table 1.1, page 2. These operators connect *statements*, such as $P$, $Q$, etc., forming new, *compound* statements $P \longrightarrow Q$, $\sim P$, $P \wedge Q$, etc. In doing so, we analyze the truth or falsity of the compound statements based upon the truth or falsity of the underlying, *component* statements $P$, $Q$, etc.[3]

We always assume a particular statement can be either true or false, but not simultaneously both.[4] We signify these possibilities by the *truth values,* T or F, respectively. Note that for $n$ independent statements $P_1, \cdots, P_n$ there are $2^n$ different combinations of T and F.[5] Thus for a single statement $P$, we have $2^1 = 2$ truth value possibilities, T or F. For two independent statements $P$ and $Q$, we have $2^2 = 4$ possible combinations of truth values: TT,TF,FT,FF, i.e., $P$ and $Q$ both true, $P$ true and $Q$ false, $P$ false and $Q$ true, or $P$ and $Q$ both false. For three statements $P, Q, R$, the possibilities are $2^3 = 8$-fold. To list exhaustively all possible orders, we will employ a *lexicographical order,* as shown in Figure 1.1. If there are $n \geq 2$ independent statements, then for the first we write half $(2^{n-1})$ T's and the same number of F's. For the next statement we write half $(2^{n-2})$ T's, and the same number of F's, and then repeat. If there is a third, we simply alternate T's with F's twice as fast, i.e., $2^{n-3}$ T's, as many F's, and then repeat following that pattern until we fill out $2^n$ entries. The last statement's entries are TFTF$\cdots$TF, until $2^n$ entries are made. Figure 1.1 illustrates this pattern, for $n = 1, 2, 3$

[3]In our analyses, the *component* statements will consist of single letters $P$, $Q$, and so on, and be allowed truth value T or F. *Compound* statements are not necessarily allowed either truth value, but their truth values are determined by those of the underlying component statements. For instance, we will see $P \vee (\sim P)$ can only have truth value T, and $P \wedge (\sim P)$ can have only F, while $P \longrightarrow Q$ is sometimes T, sometimes F.

[4]This is sometimes referred to as the "law of the excluded middle." It is useful in future discussions since it is often easier to prove $P$ is not false (i.e., $(\sim P)$ is false) than to prove $P$ is true.

[5]This is a simple counting principle. For another example suppose we have four shirts and three pairs of pants, and we want to know how many different combinations of these we can wear, assuming we will wear exactly one shirt and one pair of pants. Since we can wear any of the shirts with any of the pants, the choices—for counting purposes—are independent. We have four choices of shirts, and for each of those we have three choices of pants. It is not difficult to see that we have $4 \cdot 3 = 12$ possible combinations to choose from.

If we also include two choices of belts, and assume we will wear exactly one belt, then we have $4 \cdot 3 \cdot 2 = 24$ possible combinations of shirt, pants, and belt.

Here, there are 2 choices for the truth values (T or F) of each of the $P_1, \cdots, P_n$, so there are $2^n$ possible truth-value combinations.

Whole textbooks are written regarding this and other counting principles, but in this text we will only encounter a few. For undergraduates, some of these principles are often found embedded in probability courses or courses relying upon probability such as genetics, or in combinatorics which appears especially in computer science and electrical engineering.

| $P$ | $\sim P$ |
|---|---|
| T | F |
| F | T |

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $P \rightarrow Q$ | $P \leftrightarrow Q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T | F |
| F | F | F | F | T | T |

**Table 1.2:** The basic logic operations defined for all possible truth values of their arguments

### 1.1.2 The Logic Operations

The basic five logic operations we will use in this text are given in Table 1.2 for every possible truth value of underlying component statements. We say that the operation $\sim$ takes one *argument* (not to be confused with the colloquial meaning of the term), that argument being $P$ in the table above. The other operators $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ each take two arguments, which for the table above we dub $P$ and $Q$.

We begin with the logical negation $\sim$, which is a unary operation, i.e., acting on one (possibly compound) statement. For example consider the statement $\sim P$, usually read "not $P$." This is the negation of the statement $P$. Of course $\sim P$ is not independent of $P$, but its truth value is based upon that of $P$; stating that $\sim P$ is true is the same as stating that $P$ is false, and stating that $\sim P$ is false is the same as stating that $P$ is true. We can completely describe the relationship between $P$ and $\sim P$ in the following truth table diagram:[6]

| $P$ | $\sim P$ |
|---|---|
| T | F |
| F | T |

This also completely describes the action of the operation $\sim$: it takes a statement with truth value T and returns a statement with truth value F, and vice-versa. For an English example, if we define a statement $P$ by

$$P: \ \underline{\text{I will go to the store,}}$$

then the resultant statement for $\sim P$ is simply

$$\sim P: \ \text{I will not go to the store.}$$

We can also read $\sim$ as "it is not the case that," so our example above could read

$$\sim P: \ \text{It is not the case that I will go to the store.}$$

Even in an English example as above, it is not difficult to see that $P$ is true exactly when $\sim P$ is false, and $P$ is false exactly when $\sim P$ is true. For truth value computations, we can summarize the action of $\sim$ as follows:

$\sim$ *switches the truth value of the statement on which it operates, from* T *to* F *or from* F *to* T.

It will be interesting to see how $\sim$ operates on compound statements as we proceed. How it interacts "with itself" is rather straightforward. Indeed, it is not difficult to see that the statement $\sim(\sim P)$ is the same as the statement $P$. We might read

$$\sim(\sim P): \ \text{It is not the case that it is not the case that I will go to the store.}$$

[6]We will always use double lines to separate the independent component statements $P$, $Q$, etc., from compound statements based upon them.

Perhaps a better English translation of $\sim(\sim P)$ here would be, "it is not the case that I will not go to the store," which clearly states that I will go to the store, i.e., $P$. In the next section we will look at ways to *calculate* when two logic statements in fact mean the same thing, such as $P$ and $\sim(\sim P)$.[7]

We next turn our attention to the binary operation $\wedge$. This is called the logical *conjunction*, or just simply *and*: the statement $P \wedge Q$ is usually read "$P$ and $Q$." This compound statement $P \wedge Q$ is true exactly when both $P$ and $Q$ are true, and false if a component statement is false. Thus its truth table is given by the following:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

As an operation, $\wedge$ returns T if both statements it connects have truth value T, and returns F otherwise, i.e., if either of the statements connected by $\wedge$ is false.

**Example 1.1.1** *Suppose we set $P$ and $Q$ to be the statements*

$$P: \textit{I will eat pizza,}$$
$$Q: \textit{I will drink soda.}$$

*Connecting these with $\wedge$ gives*

$$P \wedge Q: \textit{ I will eat pizza and I will drink soda.}$$

*This is true exactly when I do both, eat pizza and drink soda, and is false if I fail to do one, or the other, or both.*

Next we look at the binary operation $\vee$, called the logical *disjunction*, or simply *or*. The statement $P \vee Q$ is usually read "$P$ or $Q$." For $P \vee Q$ to be true we only need one of the underlying component statements to be true; for $P \vee Q$ to be false we need both $P$ and $Q$ to be false. The truth table for $P \vee Q$ is thus as follows:

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

It is important to note that $P \vee Q$ is not an *exclusive or*,[8] so we still take $P \vee Q$ to be true for the case that both $P$ and $Q$ are true. At times it is not interpreted this way in spoken English, but our standard for a statement being false, i.e., having truth value F, is that it is in fact contradicted. If we state "$P$ or $Q$," to a logician we are only taken to be lying if both $P$ and $Q$ are false.

[7] It will be taken for granted throughout the text that the reader has some familiarity with the use of parentheses ( ), brackets [ ], and similar devices for grouping quantities—logical, numerical, or otherwise—to be treated as single quantities. For instance, $\sim(\sim P)$ means that the "outer" (or first) $\sim$ will operate on the statement $\sim P$, treated as a single, albeit "compound" statement. Thus we first find $\sim P$, and then its logical negation is $\sim(\sim P)$. This type of device is used throughout the chapter and the rest of the text.

[8] The case where we have $P$ or $Q$ but not both is called an *exclusive or*. Computer scientists and electrical engineers know this as **XOR**. For our purposes the *inclusive* or $\vee$ will suffice, and anyhow is much simpler to deal with computationally in symbolic logic manipulations, though **XOR** will appear in the exercises.

**Example 1.1.2** *For the $P$ and $Q$ from the previous example, we have*

$$P \vee Q: \text{ I will eat pizza or I will drink soda.}$$

*Again, this is still true if I do both, eat pizza and drink soda, or just do one of these; it is sufficient that one be true, but it is not contradicted if both are true. Note that this is false exactly when both $P$ and $Q$ are false, i.e., for the case that I do not eat pizza and do not drink soda.*

Sometimes in spoken English the above example of $P \vee Q$ would be considered false if I did both, eat the pizza and drink the soda. According to abstract logic, doing both does not technically make the speaker a liar. For many reasons, symbolic logic defines the operation $\vee$ to be inclusive, so that $P \vee Q$ is considered true in the case in which $P$ and $Q$ are both true.[9]

Next we consider $\longrightarrow$. Arguably the most common and therefore important logic statements in mathematics are of the form $P \longrightarrow Q$, read "$P$ implies $Q$" or "if $P$ then $Q$." These are also the most misunderstood by novice mathematics students, and so we will discuss them at length. As before, a truth table summarizes the action of this (binary) operation:

| $P$ | $Q$ | $P \longrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Note that the only circumstance we take $P \longrightarrow Q$ to be false is when $P$ is true, but $Q$ is false. As before, our standard for falsity is when the statement is actually contradicted, and that can be seen to be exactly when we have the truth of the *antecedent* $P$, but not of the *consequent* $Q$. In particular, if $P$ is false, then $P \longrightarrow Q$ cannot be contradicted, so we take those two cases to be true, dubbing $P \longrightarrow Q$ *vacuously true* for those two cases where $P$ is false.

In summary, the connection $\longrightarrow$ returns T for all cases *except* when the first statement is true, but the second is false.

The importance of the implication extends beyond mathematics and into philosophy and other studies. Because of its ubiquity, logical implication has several syntaxes which all mean the same to a logician. It is interesting to compare the various phrases, but first we will look at an example in the same spirit as we had for $\sim$, $\wedge$ and $\vee$.

**Example 1.1.3** *For the $P, Q$ in the previous examples, we have*

$$P \longrightarrow Q: \text{ If I will eat pizza then I will drink soda.}$$

*It is useful to see when this is clearly false: when $P$ is true but $Q$ is false, which for these $P, Q$ would be the case that I eat pizza but do not drink soda. In fact, it is important that that is the* **only** *case in which we consider $P \longrightarrow Q$ to be false. In particular, if $P$ is false, then $P \longrightarrow Q$ is vacuously true. The idea is that if I do not eat pizza, then whether or not I drink soda I do not contradict the stating, "If I will eat pizza then I will drink soda."*

[9] To see how English understanding is context-driven, consider the following situations. First, suppose a parent tells the child to "clean the bedroom or the garage" before dinner. If the child does both, the parent will likely take the request to be fulfilled. Next, suppose instead that parent tells the child to "take a cookie or a brownie" after dinner, and the child takes one of each. In this second context the parent may have a very different understanding of the child's compliance to the parental instructions. To a logician (and perhaps to any self-respecting smart aleck) $\vee$ must be context-independent.

There are several English phrases which mean $P \longrightarrow Q$. Below are five equivalent ways to write the corresponding English version of $P \longrightarrow Q$ for the $P, Q$ in the examples. (That the fourth and fifth versions are equivalent will be proved in the next section.)

1. My eating pizza implies my drinking soda ($P$ implies $Q$).
2. If I will eat pizza then I will drink soda (if $P$ then $Q$).
3. I will eat pizza only if I will drink soda ($P$ only if $Q$).
4. I will drink soda or I will not eat pizza ($Q$ or not $P$).
5. If I will not drink soda, then I will not eat pizza (if not $Q$ then not $P$).

These five ways of stating $P \longrightarrow Q$ might not all be immediately obvious, and so are worth reflection and eventual commitment to memory. Two other common—and rather elegant—ways of stating the same thing are given below in the abstract:

6. My drinking soda is necessary for my eating pizza ($Q$ is *necessary* for $P$).
7. My eating pizza is sufficient for my drinking soda ($P$ is *sufficient* for $Q$).

The kind of diction in 6 and 7 is very common in philosophical as well as mathematical discussions. We will return to implication after next discussing bi-implication, since a very common mistake for novice mathematics students is to confuse the two.

The bi-implication is denoted $P \longleftrightarrow Q$, and often read "$P$ if and only if $Q$." This is sometimes also abbreviated "$P$ iff $Q$". It states that $P$ implies $Q$ and $Q$ implies $P$ simultaneously. Thus truth of $P$ gives truth of $Q$, while truth of $Q$ would give truth of $P$. Furthermore, if $P$ is false, then so must be $Q$, because $Q$ being true would have forced $P$ to be true as well. Similarly $Q$ false would imply $P$ false (since if $P$ were instead true, so would be $Q$). The truth table for the bi-implication is the following:

| $P$ | $Q$ | $P \longleftrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

An important, alternative way to describe the operation $\longleftrightarrow$ is to note that $P \longleftrightarrow Q$ is true exactly when $P$ and $Q$ have the same truth values (TT or FF). Thus the connective $\longleftrightarrow$ can be used to detect when the connected statements' truth values match, and when they do not. This will be crucial in the next section.

**Example 1.1.4** *Consider the statement $P \longleftrightarrow Q$ for our earlier $P$ and $Q$, for which we have*

$$P \longleftrightarrow Q: \text{ I will eat pizza if and only if I will drink soda.}$$

*This is the idea that I can not have one without the other: if I have the pizza, I must also have the soda ("only if"),* **and** *I will have the pizza if I have the soda ("if"). This is false for the cases that I have one but not the other. Importantly it is not false if I have neither.*

In fact a bi-implication $P \longleftrightarrow Q$ is well-named as such since it is actually the same as $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$. (The proof of this fact is given in the next section.) Note that we can switch the order of statements connected by $\wedge$ (and), so we can instead write $(Q \longrightarrow P) \wedge (P \longrightarrow Q)$, i.e., $Q \longleftrightarrow P$. In prose we can write "$P$ is necessary and sufficient for $Q$," for $P \longleftrightarrow Q$, which is then the same as "$Q$ is necessary and sufficient for $P$," i.e., $Q \longleftrightarrow P$.

At this point we will make a few more observations concerning the differences between the English and formal logic uses of terms in common. The cases below illustrate how casual English users are often unclear about when "if," "only if," or "if and only if" are meant in both speaking and listening. Again, mathematics requires absolute precision in these things.

The first difference involves the phrase "only if." This is often misunderstood to mean "if and only if" in everyday speech. When we combine the two words "only if," the standard logic meaning is not the same as "if" modified by the adverb "only." Taken together, the words "only if" have a different, but precise meaning in logic. Consider the following statements:

- You can drive that car only if there is gasoline in the tank.
- You can drive that car only if there is air in the tires.
- You can drive that car only if the ignition system is working.

Clearly it is not the case that you can drive that car if and only if there is gasoline in the tank, since the gasoline is *necessary but not sufficient* for running the car; you also need the air, ignition, etc., or the car still will not drive regardless of the state of the gasoline tank. Similarly a father telling his teenaged child, "you can go out with your friends only if your homework is finished" might justifiably find another reason to keep the child from joining the friends even after the homework is done. (Sudden severe weather, inappropriate activities planned, mechanical problems, and several of other reasons quickly come to mind.) Note that these are all mathematical implications $\longrightarrow$: that you can drive the car would *imply* there is gas, air and ignition; that the child can go out *implies* that the homework is done. One has to be careful not to read bi-implications (if and only if) into any of these statements which are only implications.

The other difference deals with another way to state implications: if/then. This is also often misunderstood to mean if and only if. Consider the colloquial English statements:[10]

(a) If it stops raining, I'll go to the store.

(b) If I win the lottery, I'll buy a new car.

Unfortunately the "if" in statement (a) might be intended to mean "if and only if." Thus by stating (a) the speaker leads the listener to believe he will definitely go to the store if it stops raining, but also that he will go to the store *only if* it stops raining (and thus *will not* go if it does not stop raining). To the strict logician (a) is not violated in the case it does not stop raining, but the speaker still goes to the store. Recall that in such a case (a) is vacuously true.

On the other hand, it seems somewhat more likely (b) is understood the same by the logician and the casual user of English; though we are tempted to understand the speaker to mean if and only if, upon reflection we would not consider him a liar for buying the car without having first won the lottery.[11]

In both (a) and (b) the personalities and shared experiences of the speaker and listener will likely play roles in what was meant by the speaker and what was understood by the listener. In mathematics we cannot have this kind of subjectivity.

[10] This is related by Steven Zucker, Ph.D. from Johns Hopkins University, writing in the appendix of Steven Krantz's *How to Teach Mathematics*, second edition, American Mathematical Society, 1999.

[11] In everyday English, context may be important to our interpretations. For instance, if the first person were asked, "Will you go to the store," we might interpret (a) as an if and only if. For the second person, if he were asked, "what will you do if you win the lottery," then (b) might be interpreted as an "if," while if he were instead asked, "will you buy a new car," this answer might be interpreted as an "only if."

### 1.1.3 Constructing Further Truth Tables

Here we look at truth tables of more complicated compound statements. To do so, we first list the underlying component statements $P$, $Q$, and so on in lexicographical order. We then proceed—working "inside-out" and step by step—to construct the resulting truth values of the desired compound statement for each possible truth value combinations of the component statements.

It will be necessary to recall the actions of each of the operations introduced earlier. These are completely summarized by their truth tables in the previous subsection, but we can summarize the actions in words:

$\sim$ changes T to F, and F to T.

$\wedge$ returns F unless both statements it connects are true, in which case it returns T.

$\vee$ returns T if either statements it connects are true, and F exactly when both statements are false.

$\longrightarrow$ returns T except when the first statement is true and the second false. In particular, if the first statement is false, then this returns T (vacuously).

$\longleftrightarrow$ returns T if truth values of both statements match, and F if they differ.

**Example 1.1.5** *Construct a truth table for* $\sim(P \longrightarrow Q)$.

*Solution: The underlying component statements are $P$ and $Q$, so we first list these, and then their possible truth value combinations in lexicographical order. In order to construct the resulting truth table values for $\sim(P \longrightarrow Q)$, we build this statement one step at a time with the operations, in an "inside-out" fashion. By this we mean that we write the truth table column for $P \longrightarrow Q$, and then apply the negation to get the truth table column for $\sim(P \longrightarrow Q)$:*

| $P$ | $Q$ | $P \longrightarrow Q$ | $\sim(P \longrightarrow Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |

*This reflects a fact we had before: that we have $\sim(P \longrightarrow Q)$ true—i.e., we have $P \longrightarrow Q$ false—exactly when we have $P$ true but $Q$ false.*

Note that in the example above the third column, which represents $P \longrightarrow Q$, essentially connects the statements represented by the first and second columns with the connective $\longrightarrow$, while the last column applied the operation $\sim$ to the statement represented by that third column. Thus the example reads easily from left to right without interruption. It is not always possible (or easiest) to do so; often we will add a column connecting statements from previous columns which are some distance from where we want to place our new column, though our style here will always have our final column representing the desired compound statement.

**Example 1.1.6** *Compute the truth table for* $(P \vee Q) \longrightarrow (P \wedge Q)$.

*Solution: Our "inside-out" strategy is still the same. Here we list $P$ and $Q$, construct $P \vee Q$ and $P \wedge Q$ respectively, and the connect these with $\longrightarrow$:*

| $P$ | $Q$ | $P \vee Q$ | $P \wedge Q$ | $(P \vee Q) \longrightarrow (P \wedge Q)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

Some texts refer to the $\longrightarrow$ above as the "major connective," since ultimately the statement $(P \vee Q) \longrightarrow (P \wedge Q)$ is an implication, albeit connecting two already-compound statements $(P \vee Q)$ and $(P \wedge Q)$. Thus "major connective" can be seen as referring to the last operator whose action was computed in making the truth table for the statement as a whole. (In the previous example, $\sim$ would be the major connective, though we do not refer to unitary operators as "connectives.")

After constructing such a truth table step-by-step, it is also instructive to step back and examine the result. In particular, it is always useful to see which circumstances render the whole statement false, which here are the second and third combinations. In those, we have (the antecedent) $P \vee Q$ true since one of the $P$, $Q$ is true, but (the consequent) $P \wedge Q$ is not true, since $P$ and $Q$ are not both true.

**Example 1.1.7** *Construct the truth table for $P \longrightarrow [\sim (Q \vee R)]$.*

*Solution Here we need $2^3 = 8$ different combinations of truth values for the underlying component statements $P$, $Q$ and $R$. Once we list these combinations in lexicographical order, we then compute $Q \vee R$, $\sim (Q \vee R)$, and then compute $P \longrightarrow [\sim (Q \vee R)]$, in essence computing the major connective $\longrightarrow$.*

| $P$ | $Q$ | $R$ | $Q \vee R$ | $\sim (Q \vee R)$ | $P \longrightarrow [\sim (Q \vee R)]$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |

*Note that the last four cases are true vacuously. The cases where this is false are when we have $P$, but $\sim (Q \vee R)$ is false, i.e., when we have $P$ and $Q \vee R$ true, i.e., when $P$ is true and either $Q$ or $R$ is true.*

**Example 1.1.8** *Construct the truth table for $P \wedge (Q \vee R)$.*

*Solution: Again we need $2^3 = 8$ rows. The column for $P$ is repeated, though this is not necessary or always desirable (see the above example), but here was done so that the last column represents the major connective operating on the two columns immediately preceding it.*

| $P$ | $Q$ | $R$ | $P$ | $Q \vee R$ | $P \wedge (Q \vee R)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ |

In fact it is not difficult to spot which entries in the final column should have value T, since what was required was that $P$ be true, and at least one of the $Q$ or $R$ also be true. Later we will see that this has exactly the same truth value, in all circumstances, as $(P \wedge Q) \vee (P \wedge R)$, which is true if we have both $P$ and $Q$ true, or we have both $P$ and $R$ true. That these two compound statements, $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ basically state the same thing will be explored in Section 1.2, as will other "logical equivalences."

### 1.1.4 Tautologies and Contradictions, A First Look

Two very important classes of compound statements are those which form tautologies, and those which form contradictions. As we will see throughout the text, the tautologies loom especially large in our study and use of logic. We will study both tautologies and contradictions further in the next section. Here we introduce the concepts and begin to develop an intuition for these types of statements. We begin with the definitions and most obvious examples.

**Definition 1.1.1** *A compound statement formed by the component statements* $P_1, P_2, \cdots, P_n$ *is called a* **tautology** *iff its truth table column consists entirely of entries with truth value* T *for each of the* $2^n$ *possible truth value combinations (*T *and* F*) of the component statements.*

**Definition 1.1.2** *A compound statement formed by the component statements* $P_1, P_2, \cdots, P_n$ *is called a* **contradiction** *iff its truth table column consists entirely of entries with truth value* F *for each of the* $2^n$ *possible truth value combinations (*T *and* F*) of the component statements.*

**Example 1.1.9** *Consider the statement* $P \vee (\sim P)$*, which is a tautology:*

| $P$ | $\sim P$ | $P \vee (\sim P)$ |
|---|---|---|
| T | F | T |
| F | T | T |

**Example 1.1.10** *Next consider the statement* $P \wedge (\sim P)$*, which is a contradiction:*

| $P$ | $\sim P$ | $P \wedge (\sim P)$ |
|---|---|---|
| T | F | F |
| F | T | F |

We see that the statement $P \vee (\sim P)$ is always true, whereas $P \wedge (\sim P)$ is always false. There are other interesting tautologies, as well as other interesting contradictions. For the moment let us concentrate on the tautologies.

That the statement $P \vee (\sim P)$ is a tautology—especially when a particular example is examined—should be obvious when we consider what the statement says: $P$ is true or $(\sim P)$ is true. If $P$ is the statement that I will eat pizza, then we get the always true statement

$$P \vee (\sim P) : \text{I will eat pizza or I will not eat pizza.}$$

In some contexts, tautologies seem to provide no useful information. Indeed, there are times in formal speech that declaring a statement to be a tautology is meant to be demeaning. However, we will see that there are many nontrivial tautologies, and it can be quite useful to recognize complex statements which are always true.[12] For the moment we will look at the most basic of tautologies. For instance, the next tautology is obvious if we can read and understand its symbolic representation.

**Example 1.1.11** $(P \wedge Q) \longrightarrow P$ *is a tautology:*

| $P$ | $Q$ | $P \wedge Q$ | $P$ | $(P \wedge Q) \longrightarrow P$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | F | F | T |
| F | F | F | F | T |

[12] In fact, we will essentially devote the next whole section to tautologies, though we use different terms there.

*Note that the last three cases were true vacuously.*

A simple English example shows how the above is a tautology. If we take $P$ as before, and $Q$ as the statement, "I will drink soda," then $(P \wedge Q) \longrightarrow P$ becomes, "If I will eat pizza and drink soda then I will eat pizza." Looking at it abstractly, if we have both $P$ and $Q$ true, then we have $P$ true. Note that we cannot replace the implication $\longrightarrow$ with a bi-implication $\longleftrightarrow$.

In the next section we will be very much interested in tautologies in which the major connective is the bi-implication $\longleftrightarrow$. In fact we will develop a variation of the notation for just those cases. The following tautology is one such example.

**Example 1.1.12** *Show that* $[\sim (P \vee Q)] \longleftrightarrow [(\sim P) \wedge (\sim Q)]$ *is a tautology.*[13]

<u>Solution</u>:

| $P$ | $Q$ | $P \vee Q$ | $\sim (P \vee Q)$ | $\sim P$ | $\sim Q$ | $(\sim P) \wedge (\sim Q)$ | $[\sim (P \vee Q)] \longleftrightarrow [(\sim P) \wedge (\sim Q)]$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

Later we will get into the habit of just pointing out how the two columns representing, say, $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ have the same truth values, so when connected by $\longleftrightarrow$ we get a tautology. That will be more convenient, as the entire statement does not fit easily into a relatively narrow truth table column heading.

With reflection the various tautologies and contradictions become intuitive, and easy to identify. (For the above example consider the discussion of when $P \vee Q$ is false, as in Example 1.1.1, page 5.) However, not all things which appear to be contradictions are in fact contradictions. At the heart of the problem in such examples is usually the nature of the implication operation $\longrightarrow$. Consider the following:

**Example 1.1.13** *Write a truth table for* $P \longrightarrow (\sim P)$ *to demonstrate that it is* **not** *a contradiction.*

<u>Solution</u>: *Note that there is only one independent statement* $P$*, so we need only* $2^1 = 2$ *rows.*

| $P$ | $\sim P$ | $P \longrightarrow (\sim P)$ |
|---|---|---|
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |

Note that the second case is true vacuously. It is also interesting to note that the statement $P \longrightarrow (\sim P)$ has the same truth values as $\sim P$, which can be interpreted as saying $\sim P$ is the same as $P \longrightarrow (\sim P)$. That is worth pondering, but for the moment we will not elaborate.

It is perhaps easier to spot contradictions which do not involve the implication. For instance, $P \longleftrightarrow (\sim P)$ is a contradiction, but the demonstration of that (by truth tables) is left to the reader. (Simply replace $\longrightarrow$ with $\longleftrightarrow$ in the above truth table.)

[13] Some texts employ a strict hierarchy of "precedence" or "order of operations" on logic operations. It is akin to arithmetic, where $4 \cdot 5^2 + 3/5$ has us computing $5^2$, multiplying that by 4, separately computing $3/5$, and then adding our two results. With grouping symbols one might write $\left[4\left(5^2\right)\right] + [3/5]$, but through conventions designed for convenience the grouping is understood in the original expression. For our example above, some texts will simply write

$$\sim (P \vee Q) \longleftrightarrow \sim P \wedge \sim Q,$$

understanding the precedence to be $\sim$, then $\wedge$ and $\vee$, and then $\longrightarrow$ and $\longleftrightarrow$ in the order of appearance. Thus we need the first parentheses to override the precedence of the $\sim$, else we would interpret $\sim P \vee Q$ to mean $(\sim P) \vee Q$. As this text is not for a course in logic *per se*, we will continue to use grouping symbols rather than spend the effort to develop and practice a procedure for precedence. (Besides, the authors find the claerly grouped statements easier and more pleasing to read and write.)

# Exercises

1. A very useful way to learn the nuances of the logic operations is to consider when their compound statements are false. For each of the following compound statements, discuss all possible circumstances in which the given statement is false.

   For example, $P \longleftrightarrow Q$ is false exactly when $P$ is true and $Q$ false, or $Q$ true and $P$ false.

   (a) $\sim P$

   (b) $P \wedge Q$

   (c) $P \vee Q$

   (d) $P \longrightarrow Q$

   (e) $P \longleftrightarrow Q$

   (f) $P \longrightarrow (\sim Q)$.

2. Repeat 1(a)–(f) above, except using truth tables for each to answer the question of when each statement is false. Compare and reconcile your answers to Exercise 1 above.

3. Consider the statement

   $$(\sim Q) \longrightarrow (\sim P).$$

   (a) When is it false?

   (b) Now consider $P \longrightarrow Q$. When is it false?

   (c) Do you believe these two compound statements mean the same thing?

   (d) Construct the truth table for the statement $(\sim Q) \longrightarrow (\sim P)$. Then revisit your answer to (c).

4. Construct the truth table for $P$ **XOR** $Q$. (See Footnote 8, page 5.)

5. Construct the truth table for the statement $\sim (P \longleftrightarrow Q)$. Compare your answer to the previous exercise.

6. Construct truth tables for the following statements:

   (a) $(\sim P) \longleftrightarrow (\sim Q)$ (Compare to $P \longleftrightarrow Q$.)

   (b) $[P \vee (\sim Q)] \longrightarrow P$

   (c) $\sim [P \wedge (Q \vee R)]$

7. Find six other English statements which are equivalent to the statement,

   > "You can go out with your friends only if your homework is finished."

   (See page 7. Some of your answers may seem very formal.)

8. Construct truth tables for the following statements.

   (a) $\sim (P \wedge Q)$

   (b) $P \vee (Q \wedge R)$

   (c) $P \vee (Q \vee R)$

   (d) $(P \vee Q) \vee R$ (Compare to the previous statement.)

   (e) $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$

9. Decide which are tautologies, which are contradictions, and which are neither. Try to decide using intuition, and then check with truth tables.

   (a) $P \longrightarrow P$

   (b) $P \longleftrightarrow P$

   (c) $P \vee (\sim P)$

   (d) $P \wedge (\sim P)$

   (e) $P \longleftrightarrow (\sim P)$

   (f) $P \longrightarrow (\sim P)$

   (g) $((P) \wedge (\sim P)) \longrightarrow Q$

   (h) $(P \longrightarrow (\sim P)) \longrightarrow (\sim P)$

   (i) $(P \wedge Q) \longrightarrow P$

   (j) $(P \vee Q) \longrightarrow P$

   (k) $P \longrightarrow (P \wedge Q)$

   (l) $P \longrightarrow (P \vee Q)$

10. Some confuse implication $\longrightarrow$ with causation, interpreting $P \longrightarrow Q$ as "$P$ causes $Q$." However, the implication is in fact weaker than the layman's concept of causation. Answer the following:

(a) Show that $(P \longrightarrow Q) \vee (Q \longrightarrow P)$ is a tautology.

(b) Explain why, replacing $\longrightarrow$ with the phrase "causes" clearly does not give us a tautology.

(c) On the other hand, if $P$ being true **causes** $Q$ to be true, can we say $P \longrightarrow Q$ is true?

11. Write the lexicographical ordering of the possible truth value combinations for four statements $P, Q, R, S$.

## 1.2 Valid Logical Equivalences as Tautologies

### 1.2.1 The Idea, and Definition, of Logical Equivalence

In lay terms, two statements are logically equivalent when they say the same thing, albeit perhaps in different ways. To a mathematician, two statements are called logically equivalent when they will always be simultaneously true or simultaneously false. To see that these notions are compatible, consider an example of a man named John N. Smith who lives alone at 12345 North Fictional Avenue in Miami, Florida, and has a United States Social Security number 987-65-4325.[14] Of course there should be exactly one person with a given Social Security number. Hence, when we ask any person the questions, "are you John N. Smith of 12345 North Fictional Avenue in Miami, Florida?" and "is your U.S. Social Security number 987-65-4325?" we would be in essence asking the same question in both cases. Indeed, the answers to these two questions would always be both yes, or both no, so the statements "you are John N. Smith of 12345 North Fictional Avenue in Miami, Florida," and "your U.S. Social Security number is 987-65-4325," are logically equivalent. The notation we would use is the following:

you are John N. Smith of 12345 North Fictional Avenue in Miami, Florida
$\iff$ your U.S. Social Security number is 987-65-4325.

The motivation for the notation " $\iff$ " will be explained shortly.

On a more abstract note, consider the statements $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$. Below we compute both of these compound statements' truth values in one table:

| $P$ | $Q$ | $P \vee Q$ | $\sim (P \vee Q)$ | $\sim P$ | $\sim Q$ | $(\sim P) \wedge (\sim Q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

the same

We see that these two statements are both true or both false, under any of the $2^2 = 4$ possible circumstances, those being the possible truth value combinations of the underlying, independent component statements $P$ and $Q$. Thus the statements $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ are indeed logically equivalent in the sense of always having the same truth value. Having established this, we would write

$$\sim (P \vee Q) \iff (\sim P) \wedge (\sim Q).$$

Note that in logic, this symbol " $\iff$ " is similar to the symbol "=" in algebra and elsewhere.[15] There are a couple of ways it is read out loud, which we will consider momentarily. For now we take the occasion to list the formal definition of logical equivalence:

**Definition 1.2.1** *Given $n$ independent statements $P_1, \cdots, P_n$, and two statements $R, S$ which are compound statements of the $P_1, \cdots, P_n$, we say that $R$ and $S$ are* **logically equivalent***, which we then denote $R \iff S$, if and only if their truth table columns have the same entries for each of the $2^n$ distinct combinations of truth values for the $P_1, \cdots, P_n$. When $R$ and $S$ are logically equivalent, we will also call $R \iff S$ a* **valid equivalence***.*

[14] This is all, of course, fictitious.

[15] Some texts use the symbol "$\equiv$" for logical equivalence. However there is another standard use for this symbol, and so we will reserve that symbol for that use later in the text.

Again, this is consistent with the idea that to say statements $R$ and $S$ are logically equivalent is to say that, under any circumstances, they are both true or both false, so that asking if $R$ is true is—functionally—exactly the same as asking if $S$ is true. (Recall our example of John N. Smith's Social Security number.)

Note that if two statements' truth values always match, then connecting them with $\longleftrightarrow$ yields a tautology. Indeed, the bi-implication yields T if the connected statements have the same truth value, and $F$ otherwise. Since two logically equivalent statements will have matching truth values in all cases, connecting with $\longleftrightarrow$ will always yield T, and we will have a tautology. On the other hand, if connecting two statements with $\longleftrightarrow$ forms a tautology, then the connected statements must have always-matching truth values, and thus be equivalent. This argument yields our first theorem:[16]

**Theorem 1.2.1** *Suppose $R$ and $S$ are compound statements of $P_1 \cdots, P_n$. Then $R$ and $S$ are logically equivalent if and only if $R \longleftrightarrow S$ is a tautology.*

The theorem above gives us the motivation behind the notation $\iff$. Assuming $R$ and $S$ are compound statements built upon component statements $P_1 \cdots, P_n$, then

$$R \iff S \qquad \text{means that} \qquad R \longleftrightarrow S \text{ is a tautology.} \tag{1.1}$$

To be clear, when we write $R \longleftrightarrow S$ we understand that this might have truth value T or F, i.e., it might be true or false. However, when we write $R \iff S$, we mean that $R \longleftrightarrow S$ is always true (i.e., a tautology), which partially explains why we call $R \iff S$ a *valid equivalence*.[17]

To prove $R \iff S$, we could (but usually will not) construct $R \longleftrightarrow S$, and show that it is a tautology. We do so below to prove

$$\underbrace{\sim (P \vee Q)}_{\text{“}R\text{”}} \iff \underbrace{(\sim P) \wedge (\sim Q)}_{\text{“}S\text{”}}.$$

| $P$ | $Q$ | $P \vee Q$ | $\overbrace{\sim (P \vee Q)}^{R}$ | $\sim P$ | $\sim Q$ | $\overbrace{(\sim P) \wedge (\sim Q)}^{S}$ | $\overbrace{[\sim (P \vee Q)] \longleftrightarrow [(\sim P) \wedge (\sim Q)]}^{R \longleftrightarrow S}$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | T |
| T | F | T | F | F | T | F | T |
| F | T | T | F | T | F | F | T |
| F | F | F | T | T | T | T | T |

However, our preferred method will be as in the previous truth table, where we simply show that the truth table columns for $R$ and $S$ have the same entries at each horizontal level, i.e., for each truth value combination of the component statements. That approach saves space and reinforces our original notion of equivalence (matching truth values). However it is still important to understand the connection between $\longleftrightarrow$ and $\iff$, as given in (1.1).

[16]A **theorem** is a fact which has been proven to be true, particularly dealing with mathematics. We will state numerous theorems in this text. Most we will prove, though occasionally we will include a theorem which is too relevant to omit, but whose proof is too technical to include in an undergraduate calculus book. Such proofs are left to courses with titles such as mathematical (or real) analysis, topology, or advanced calculus.

Some theorems are also called lemmas (or, more archaically, lemmata) when they are mostly useful as steps in larger proofs of the more interesting results. Still others are called corollaries if they are themselves interesting, but follow with very few extra steps after the underlying theorem is proved.

[17]Depending upon the author, both $R \longleftrightarrow S$ and $R \iff S$ are sometimes verbalized “$R$ is equivalent to $S$,” or “$R$ if and only if $S$.” We distinguish the cases by using the term “equivalent” for the double-lined arrow, and “if and only if” for the single-lined arrow. To help avoid confusion, we emphasize this more restrictive use of “equivalences” (denoted with $\iff$) by calling them “valid equivalences.”

### 1.2.2 Equivalences for Negations

Much of the intuition achieved from studying symbolic logic comes from examining various logical equivalences. Indeed we will make much use of these, for the theorems we use throughout the text are often stated in one form, and then used in a different, but logically equivalent form. When we prove a theorem, we may prove even a third, logically equivalent form.

The first logical equivalences we will look at here are the negations of the our basic operations. We already looked at the negations of $\sim P$ and $P \vee Q$. Below we also look at negations of $P \wedge Q$, $P \longrightarrow Q$ and $P \longleftrightarrow Q$. Historically, (1.3) and (1.4) below are called *De Morgan's Laws*, but each basic negation is important. We now list these negations.

$$\sim(\sim P) \iff P \tag{1.2}$$
$$\sim(P \vee Q) \iff (\sim P) \wedge (\sim Q) \tag{1.3}$$
$$\sim(P \wedge Q) \iff (\sim P) \vee (\sim Q) \tag{1.4}$$
$$\sim(P \longrightarrow Q) \iff P \wedge (\sim Q) \tag{1.5}$$
$$\sim(P \longleftrightarrow Q) \iff [P \wedge (\sim Q)] \vee [Q \wedge (\sim P)]. \tag{1.6}$$

Fortunately, with a well chosen perspective these are intuitive. Recall that any statement $R$ can also be read "$R$ is true," while the negation asserts the original statement is false. For example $\sim R$ can be read as the statement "$R$ is false," or a similar wording (such as "it is not the case that $R$"). Similarly the statement $\sim(P \vee Q)$ is the same as "'$P$ or $Q$' is false." With that it is not difficult to see that for $\sim(P \vee Q)$ to be true requires both that $P$ be false *and* $Q$ be false. For a specific example, consider our earlier $P$ and $Q$:

| | |
|---|---|
| $P$ : | I will eat pizza |
| $Q$ : | I will drink soda |
| $P \vee Q$ : | I will eat pizza or I will drink soda |
| $\sim(P \vee Q)$ : | It is not the case that (either) I will eat pizza or I will drink soda |
| $(\sim P) \wedge (\sim Q)$ : | It is not the case that I will eat pizza, and it is not the case that I will drink soda |

That these last two statements essentially have the same content, as stated in (1.3), should be intuitive. An actual *proof* of (1.3) is best given by truth tables, and can be found on page 15.

Next we consider (1.5). This states that $\sim(P \longrightarrow Q) \iff P \wedge (\sim Q)$. Now we can read $\sim(P \longrightarrow Q)$ as "it is not the case that $P \longrightarrow Q$," or "$P \longrightarrow Q$ is false." Recall that there was only one case for which we considered $P \longrightarrow Q$ to be false, which was the case that $P$ was true but $Q$ was false, which itself can be translated to $P \wedge (\sim Q)$. For our earlier example, the negation of the statement "if I eat pizza then I will drink soda" is the statement "I will eat pizza but (and) I will not drink soda." While this discussion is correct and may be intuitive, the actual proof (1.5) is by truth table:

| $P$ | $Q$ | $P \to Q$ | $\sim(P \to Q)$ | $P$ | $\sim Q$ | $P \wedge (\sim Q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | F | F |
| T | F | F | T | T | T | T |
| F | T | T | F | T | F | F |
| F | F | T | F | F | T | F |

the same

We leave the proof of (1.6) by truth tables to the exercises. Recall that $P \longleftrightarrow Q$ states that we have $P$ true if and only if we also have $Q$ true, which we further translated as the idea that we cannot have $P$ true without $Q$ true, and cannot have $Q$ true without $P$ true. Now $\sim (P \longleftrightarrow Q)$ is the statement that $P \longleftrightarrow Q$ is false, which means that $P$ is true and $Q$ false, or $Q$ is true and $P$ false, which taken together form the statement $[P \wedge (\sim Q)] \vee [Q \wedge (\sim P)]$, as reflected in (1.6) above. For our example $P$ and $Q$ from before, $P \longleftrightarrow Q$ is the statement "I will at pizza if and only if I will drink soda," the negation of which is "I will eat pizza and not drink soda, or I will drink soda and not eat pizza."

Another intuitive way to look at these negations is to consider the question of exactly when is someone uttering the original statement lying? For instance, if someone states $P \wedge Q$ (or some English equivalent), when are they lying? Since they stated "$P$ and $Q$," it is not difficult to see they are lying exactly when at least one of the statements $P, Q$ is false, i.e., when $P$ is false or $Q$ is false,[18] i.e., when we can truthfully state $(\sim P) \vee (\sim Q)$. That is the kind of thinking one should employ when examining (1.4), that is $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, intuitively.

### 1.2.3 Equivalent Forms of the Implication

In this subsection we examine two statements which are equivalent to $P \longrightarrow Q$. The first is more important conceptually, and the second is more important computationally. We list them both now before contemplating them further:

$$P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P) \tag{1.7}$$

$$P \longrightarrow Q \iff (\sim P) \vee Q. \tag{1.8}$$

We will combine the proofs into one truth table, where we compute $P \longrightarrow Q$, followed in turn by $(\sim Q) \longrightarrow (\sim P)$ and $(\sim P) \vee Q$.

| $P$ | $Q$ | $P \to Q$ | $\sim Q$ | $\sim P$ | $(\sim Q) \to (\sim P)$ | $\sim P$ | $Q$ | $(\sim P) \vee Q$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F | T | T |
| T | F | F | T | F | F | F | F | F |
| F | T | T | F | T | T | T | T | T |
| F | F | T | T | T | T | T | F | T |

the same

The form (1.7) is important enough that it warrants a name:

**Definition 1.2.2** *Given any implication $P \longrightarrow Q$, we call the (logically equivalent) statement $(\sim Q) \longrightarrow (\sim P)$ its* **contrapositive** *(and vice-versa, see below).*

In fact, note that the contrapositive of $(\sim Q) \longrightarrow (\sim P)$ would be $[\sim (\sim P)] \longrightarrow [\sim (\sim Q)]$, i.e., $P \longrightarrow Q$, so $P \longrightarrow Q$ and $(\sim Q) \longrightarrow (\sim P)$ are contrapositives *of each other.*

We have proved that $P \longrightarrow Q$, its contrapositive $(\sim Q) \longrightarrow (\sim P)$, and the other form $(\sim P) \vee Q$ are equivalent using the truth table above, but developing the intuition that these *should* be equivalent can require some effort. Some examples can help to clarify this.

[18] Note that here as always we use the inclusive "or," so when we write "$P$ is false or $Q$ is false," we include the case in which both $P$ and $Q$ are false. (See Footnote 8, page 5 for remarks on the exclusive "or.")

$$\begin{aligned}
P &: \text{I will eat pizza}\\
Q &: \text{I will drink soda}\\
P \longrightarrow Q &: \text{If I eat pizza, then I will drink soda}\\
(\sim Q) \longrightarrow (\sim P) &: \text{If I do not drink soda, then I will not eat pizza}\\
(\sim P) \vee Q &: \text{I will not eat pizza, or I will drink soda.}
\end{aligned}$$

Perhaps more intuition can be found when $Q$ is a more natural consequence of $P$. Consider the following $P, Q$ combination which might be used by parents communicating to their children.

$$\begin{aligned}
P &: \text{you leave your room messy}\\
Q &: \text{you get spanked}\\
P \longrightarrow Q &: \text{if you leave your room messy, then you get spanked}\\
(\sim Q) \longrightarrow (\sim P) &: \text{if you do not get spanked, then you do (did) not leave your room messy}\\
(\sim P) \vee Q &: \text{you do not leave your room messy, or you get spanked.}
\end{aligned}$$

A mathematical example could look like the following (assuming $x$ is a "real number," as discussed later in this text):

$$\begin{aligned}
P &: x = 10\\
Q &: x^2 = 100\\
P \longrightarrow Q &: \text{if } x = 10, \text{ then } x^2 = 100\\
(\sim Q) \longrightarrow (\sim P) &: \text{if } x^2 \neq 100, \text{ then } x \neq 10\\
(\sim P) \vee Q &: x \neq 10 \text{ or } x^2 = 100.
\end{aligned}$$

The contrapositive is very important because many theorems are given as implications, but are often used in their logically equivalent, contrapositive forms. However, it is equally important to avoid confusing $P \longrightarrow Q$ with either of the statements $P \longleftrightarrow Q$ or $Q \longrightarrow P$. For instance, in the second example above, the child may get spanked without leaving the room messy, as there are quite possibly other infractions which would result in a spanking. Thus leaving the room messy does not follow from being spanked, and leaving the room messy is not necessarily connected with the spanking by an "if and only if." In the last, algebraic example above, all the forms of the statement are true, but $x^2 = 100$ does not imply $x = 10$. Indeed, it is possible that $x = -10$. In fact, the correct bi-implication is $x^2 = 100 \longleftrightarrow [(x = 10) \vee (x = -10)]$.

### 1.2.4 Other Valid Equivalences

While negations and equivalent alternatives to the implication are arguably the most important of our valid logical equivalences, there are several others. Some are rather trivial, such as

$$P \wedge P \iff P \iff P \vee P. \tag{1.9}$$

Also rather easy to see are the "commutativities" of $\wedge, \vee$ and $\longleftrightarrow$:

$$P \wedge Q \iff Q \wedge P, \qquad P \vee Q \iff Q \vee P, \qquad P \longleftrightarrow Q \iff Q \longleftrightarrow P. \tag{1.10}$$

There are also associative rules. The latter was in fact a topic in the previous exercises:

$$P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R \tag{1.11}$$
$$P \vee (Q \vee R) \iff (P \vee Q) \vee R. \tag{1.12}$$

However, it is not so clear when we mix together $\vee$ and $\wedge$. In fact, these "distribute over each other" in the following ways:

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R), \tag{1.13}$$
$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R). \tag{1.14}$$

We prove the first of these distributive rules below, and leave the other for the exercises.

| $P$ | $Q$ | $R$ | $Q \vee R$ | $P \wedge (Q \vee R)$ | $P \wedge Q$ | $P \wedge R$ | $(P \wedge Q) \vee (P \wedge R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | F | F | F | F | F |
| F | T | T | T | F | F | F | F |
| F | T | F | T | F | F | F | F |
| F | F | T | T | F | F | F | F |
| F | F | F | F | F | F | F | F |

the same

To show that this is reasonable, consider the following:

$P$ : I will eat pizza;
$Q$ : I will drink cola;
$R$ : I will drink lemon-lime soda.

Then our logically equivalent statements become

$P \wedge (Q \vee R)$ : I will eat pizza, and drink cola or lemon-lime soda;
$(P \wedge Q) \vee (P \wedge R)$ : I will eat pizza and drink cola, or
I will eat pizza and drink lemon-lime soda.

Table 1.3, page 22 gives these and some further valid equivalences. It is important to be able to read these and, through reflection and the exercises, to be able to see the reasonableness of each of these. Each can be proved using truth tables.

For instance we can prove that $P \longleftrightarrow Q \iff (P \longrightarrow Q) \wedge (Q \longrightarrow P)$, justifying the choice of the double-arrow symbol $\longleftrightarrow$:

| $P$ | $Q$ | $P \longleftrightarrow Q$ | $P \longrightarrow Q$ | $Q \longrightarrow P$ | $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | T | T |

the same

This was discussed in Example 1.1.4 on page 7.

For another example of such a proof, we next demonstrate the following interesting equivalence:

$$P \longrightarrow (Q \wedge R) \iff (P \longrightarrow Q) \wedge (P \longrightarrow R)$$

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $P \longrightarrow (Q \wedge R)$ | $P \longrightarrow Q$ | $P \longrightarrow R$ | $(P \longrightarrow Q) \wedge (P \longrightarrow R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | F |
| T | F | T | F | F | F | T | F |
| T | F | F | F | F | F | F | F |
| F | T | T | T | T | T | T | T |
| F | T | F | F | T | T | T | T |
| F | F | T | F | T | T | T | T |
| F | F | F | F | T | T | T | T |

the same

This should be somewhat intuitive: if $P$ is to imply $Q \wedge R$, that should be the same as $P$ implying $Q$ *and* $P$ implying $R$. This equivalence will be (1.33), page 22. According to (1.34) below it, we can replace $\wedge$ with $\vee$ and get another valid equivalence.

Still one must be careful about declaring two statements to be equivalent. These are all ultimately intuitive, but intuition must be informed.[19] For instance, left to the exercises are some valid equivalences which may seem counter-intuitive. These are in fact left off of our Table 1.3 because they are somewhat obscure, but we include them here to illustrate that not all equivalences are transparent. Consider

$$(P \vee Q) \longrightarrow R \iff (P \longrightarrow R) \wedge (Q \longrightarrow R), \tag{1.15}$$
$$(P \wedge Q) \longrightarrow R \iff (P \longrightarrow R) \vee (Q \longrightarrow R). \tag{1.16}$$

Upon reflection one can see how these are reasonable. For instance, we can look more closely at (1.15) with the following $P, Q$ and $R$:

$P$ : I eat pizza,
$Q$ : I eat chicken,
$R$ : I drink cola.

Then the left and right sides of (1.15) become

$(P \vee Q) \longrightarrow R$ : If I eat pizza or chicken, then I drink cola
$(P \longrightarrow R) \wedge (Q \longrightarrow R)$ : If I eat pizza then I drink cola, and if I eat chicken then I drink cola.

In fact (1.16) is perhaps more difficult to see.

At the end of the chapter there will be an optional section for the reader interested in achieving a higher level of symbolic logic sophistication. That section is devoted to finding and proving valid equivalences (and implications as seen in the next section) without relying on truth tables. The technique centers on using a small number of established equivalences to rewrite compound statements into alternative, equivalent forms. With those techniques one can quickly prove (1.15) and (1.16), again without truth tables. It is akin to proving trigonometric identities, or the leap from memorizing single-digit multiplication tables and applying them to several-digit problems.

[19] It is a truism in mathematics and other fields that, while one part of learning is discovering what is true, another part is discovering what is not true, especially when the latter seems reasonable at first glance.

$$
\begin{aligned}
P \wedge P &\iff P \iff P \vee P && (1.17)\\
\sim(\sim P) &\iff P && (1.18)\\
\sim(P \vee Q) &\iff (\sim P) \wedge (\sim Q) && (1.19)\\
\sim(P \wedge Q) &\iff (\sim P) \vee (\sim Q) && (1.20)\\
\sim(P \longrightarrow Q) &\iff P \wedge (\sim Q) && (1.21)\\
\sim(P \longleftrightarrow Q) &\iff [P \wedge (\sim Q)] \vee [Q \wedge (\sim P)] && (1.22)\\
P \vee Q &\iff Q \vee P && (1.23)\\
P \wedge Q &\iff Q \wedge P && (1.24)\\
P \vee (Q \vee R) &\iff (P \vee Q) \vee R && (1.25)\\
P \wedge (Q \wedge R) &\iff (P \wedge Q) \wedge R && (1.26)\\
P \wedge (Q \vee R) &\iff (P \wedge Q) \vee (P \wedge R) && (1.27)\\
P \vee (Q \wedge R) &\iff (P \vee Q) \wedge (P \vee R) && (1.28)\\
P \longrightarrow Q &\iff (\sim P) \vee Q && (1.29)\\
P \longrightarrow Q &\iff (\sim Q) \longrightarrow (\sim P) && (1.30)\\
P \longrightarrow Q &\iff \sim [P \wedge (\sim Q)] && (1.31)\\
P \longleftrightarrow Q &\iff (\sim P) \longleftrightarrow (\sim Q) && (1.32)\\
P \longrightarrow (Q \wedge R) &\iff (P \longrightarrow Q) \wedge (P \longrightarrow R) && (1.33)\\
P \longrightarrow (Q \vee R) &\iff (P \longrightarrow Q) \vee (P \longrightarrow R) && (1.34)\\
(P \longrightarrow Q) \wedge (Q \longrightarrow P) &\iff P \longleftrightarrow Q && (1.35)\\
(P \longrightarrow Q) \wedge (Q \longrightarrow R) \wedge (R \longrightarrow P) &\iff (P \longleftrightarrow Q) \wedge (Q \longleftrightarrow R) &&\\
&\qquad \wedge (P \longleftrightarrow R) && (1.36)
\end{aligned}
$$

**Table 1.3:** Table of common valid logical equivalence.

For a glance at the process, we can look at such a proof of the equivalence of the contrapositive: $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$. To do so, we require (1.29), that $P \longrightarrow Q \iff (\sim P) \vee Q$. The proof runs as follows:

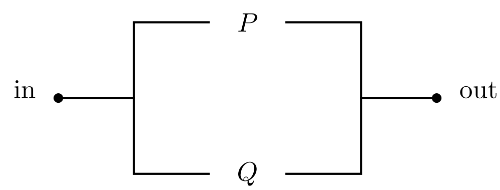
$$
\begin{aligned}
P \longrightarrow Q &\iff (\sim P) \vee Q\\
&\iff Q \vee (\sim P)\\
&\iff [\sim(\sim Q)] \vee (\sim P)\\
&\iff (\sim Q) \longrightarrow (\sim P).
\end{aligned}
$$

The first line used (1.29), the second commutativity (1.23), the third that $Q \iff \sim(\sim Q)$ (1.18), and the fourth used (1.29) again but with the part of "$P$" played by $(\sim Q)$ and the part of "$Q$" played by $(\sim P)$. This proof is not much more efficient than a truth table proof, but for (1.15) and (1.16) this technique of proofs without truth tables is much faster. However that technique assumes that the more primitive equivalences used in the proof are valid, and those are ultimately proved using truth tables. The extra section which develops such techniques, namely Section 1.6, is supplemental and not required reading for understanding sufficient symbolic logic to aid in developing the calculus. For that we need only up through Section 1.4.

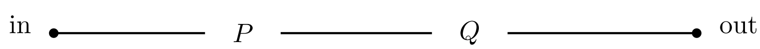
### 1.2.5 Circuits and Logic

While we will not develop this next theory deeply, it is worthwhile to consider a short introduction. The idea is that we can model compound logic statements with electrical switching circuits.[20] When current is allowed to flow across a switch, the switch is considered "on" when the statement it represents has truth value T and current can flow through the switch, and "off" and not allowing current to flow through when the truth value is F. We can decide if the compound circuit is "on" or "off" based upon whether or not current could flow from one end to the other, based on whether the compound statement has truth value T or F. The analysis can be complicated if the switches are not necessarily independent ($P$ is "on" when $\sim P$ is "off" for instance), but this approach is interesting nonetheless.

For example, the statement $P \vee Q$ is represented by a parallel circuit:




If either $P$ or $Q$ is on (T), then the current can flow from the "in" side to the "out" side of the circuit. On the other hand, we can represent $P \wedge Q$ by a series circuit:

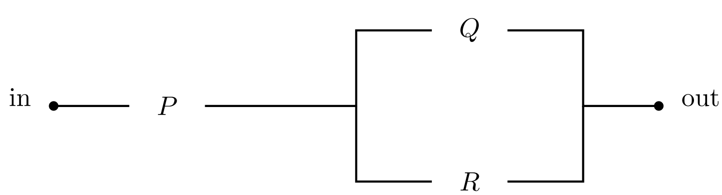


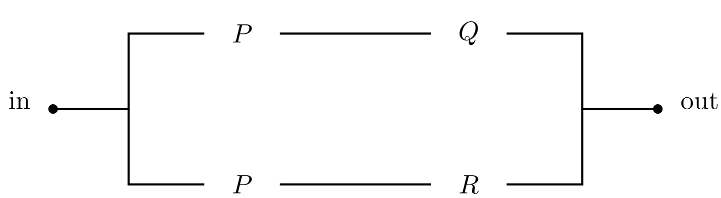

Of course $P \wedge Q$ is only true when both $P$ and $Q$ are true, and the circuit reflects this: current can flow exactly when both "switches" $P$ and $Q$ are "on."

It is interesting to see diagrams of some equivalent compound statements, illustrated as circuits. For instance, (1.27), i.e., the distributive-type equivalence

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$

can be seen as the equivalence of the two cicruits below:

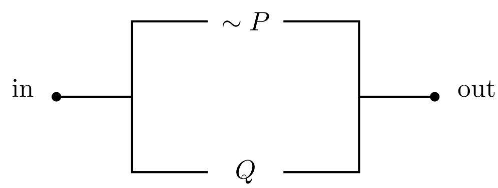



[20]Technically a *circuit* would allow current to flow from a source, through components and back to the source. Here we only show part of the possible path. We will encounter some complete circuits later in the text.

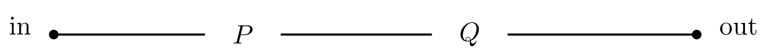
In both circuits, we must have $P$ "on," and also either $Q$ or $R$ for current to flow. Note that in the second circuit, $P$ is represented in two places, so it is either "on" in both places, or "off" in both places. Situations such as these can complicate analyses of switching circuits but this one is relatively simple.

We can also represent negations of simple statements. To represent $\sim P$ we simply put "$\sim P$" into the circuit, where it is "on" if $\sim P$ is true, i.e., if $P$ is false. This allows us to construct circuits for the implication by using (1.29), i.e., that $P \longrightarrow Q \iff (\sim P) \vee Q$:

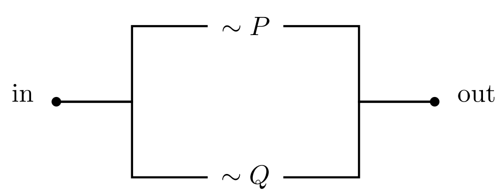



We see that the only time the circuit does *not* flow is when $P$ is true ($\sim P$ is false) and $Q$ is false, so this matches what we know of when $P \longrightarrow Q$ is false. From another perspective, if $P$ is true, then the top part of the circuit won't flow so $Q$ must be true, for the whole circuit to be "on," or "true."

When negating a whole circuit it gets even more complicated. In fact, it is arguably easier to look at the original circuit and simply note when current will *not* flow. For instance, we know $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, so we can construct $P \wedge Q$:




and note that it is *off* exactly when either $P$ is off or $Q$ is off. We then note that that is exactly when the circuit for $(\sim P) \vee (\sim Q)$ is *on*.

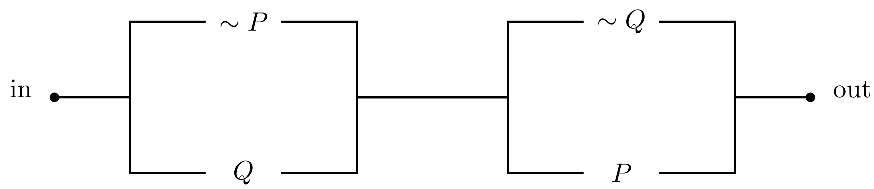



There are, in fact, electrical/mechanical means by which one can take a circuit and "negate" its truth value, for instance with relays or reverse-position switch levers, but that subject is more complicated than we wish to pursue here.

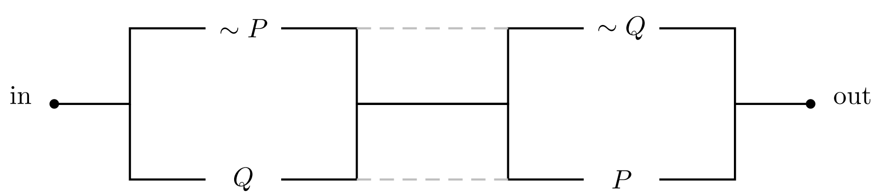
It is interesting to consider $P \longleftrightarrow Q$ as a circuit. It will be "on" if $P$ and $Q$ are both "on" or both "off," and the circuit will be "off" if $P$ and $Q$ do not match. Such a circuit is actually used commonly, such as for a room with two light switches for the same light. To construct such a circuit we note that

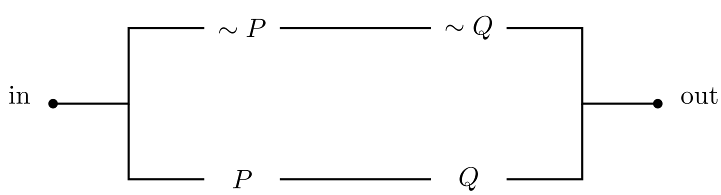
$$\begin{aligned} P \longleftrightarrow Q &\iff (P \longrightarrow Q) \wedge (Q \longrightarrow P) \\ &\iff [(\sim P) \vee Q] \wedge [(\sim Q) \vee P] \end{aligned}$$

We will use the last form to draw our diagram:




The reader is invited to study the above diagram to be convinced it represents $P \longleftrightarrow Q$, perhaps most easily in the sense that, "you can not have one ($P$ or $Q$) without the other, but you can have neither." While the above diagram does represent $P \longleftrightarrow Q$ by the more easily diagrammed $[(\sim P) \vee Q] \wedge [(\sim Q) \vee P]$, it also suggests another equivalence, since the circuits below seems to be functionally equvialent. In the first, we can add two more wires to replace the "center" wire, and also switch the $\sim Q$ and $P$, since $(\sim Q) \vee P$ is the same as $P \vee (\sim Q)$:



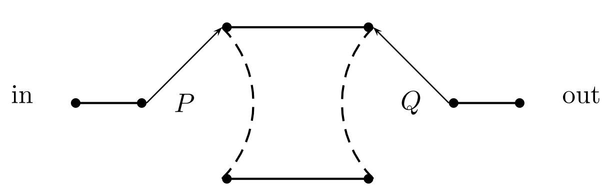




This circuit represents $[(\sim P) \wedge (\sim Q)] \vee [P \wedge Q]$, and so we have (as the reader can check)

$$P \longleftrightarrow Q \iff [(\sim P) \wedge (\sim Q)] \vee [P \wedge Q], \tag{1.37}$$

which could be added to our previous Table 1.3, page 22 of valid equivalences. It is also consistent with a more colloquial way of expressing $P \longleftrightarrow Q$, such as "neither or both."

Incidentally, the circuit above is used in applications where we wish to have two switches within a room which can both change a light (or other device) from on to off or vice versa. When switch $P$ is "on," switch $Q$ can turn the circuit on or off by matching $P$ or being its negation. Similarly when $P$ is "off." Mechanically this is accomplished with "single pole, double throw (SPDT)" switches.




In the above, the switch $P$ is in the "up" position when $P$ is 'true, and "down" when $P$ is false. Similarly with $Q$.

Because there are many possible "mechanical" diagrams for switching circuits, reading and writing such circuits is its own skill. However, for many simpler cases there is a relatively easy connection to our symbolic logic.

### 1.2.6 The Statements $\mathcal{T}$ and $\mathcal{F}$

Just as there is a need for zero in addition, we have use for a symbol representing a statement which is always true, and for another symbol representing a statement which is always false. For convenience, we will make the following definitions:

**Definition 1.2.3** *Let $\mathcal{T}$ represent any compound statement which is a* **tautology***, i.e., whose truth value is always* T*. Similarly, let $\mathcal{F}$ represent any compound statement which is a* **contradiction***, i.e., whose truth value is always* F*.*

We will assume there is a universal $\mathcal{T}$ and a universal $\mathcal{F}$, i.e., statements which are respectively true regardless of any other statements' truth values, and false regardless of any other statements' truth values. In doing so, we consider any tautology to be logically equivalent to $\mathcal{T}$, and any contradiction similarly equivalent to $\mathcal{F}$.[21]

So, for any given $P_1 \cdots, P_n$, we have that $\mathcal{T}$ is exactly that statement whose column in the truth table consists entirely of T's, and $\mathcal{F}$ is exactly that statement whose column in the truth table consists entirely of F's. For example, we can write

$$P \vee (\sim P) \iff \mathcal{T}; \tag{1.38}$$
$$P \wedge (\sim P) \iff \mathcal{F}. \tag{1.39}$$

These are easily seen by observing the truth tables.

| $P$ | $\sim P$ | $P \vee (\sim P)$ | $P \wedge (\sim P)$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

We see that $P \vee (\sim P)$ is always true, and $P \wedge (\sim P)$ is always false. Anything which is always true we will dub $\mathcal{T}$, and anything which is always false we will call $\mathcal{F}$. In the table above, the third column represents $\mathcal{T}$, and the last column represents $\mathcal{F}$.

From the definitions we can also eventually get the following.

$$P \vee \mathcal{T} \iff \mathcal{T} \tag{1.40}$$
$$P \wedge \mathcal{T} \iff P \tag{1.41}$$
$$P \vee \mathcal{F} \iff P \tag{1.42}$$
$$P \wedge \mathcal{F} \iff \mathcal{F}. \tag{1.43}$$

[21] In fact it is not difficult to see that all tautologies are logically equivalent. Consider the tautologies $P \vee (\sim P)$, $(P \longrightarrow Q) \longleftrightarrow [(\sim Q) \longrightarrow (\sim P)]$, and $R \longrightarrow R$. A truth table for all three must contain independent component statements $P, Q, R$, and the abridged version of the table would look like

| $P$ | $Q$ | $R$ | $P \vee (\sim P)$ | $(P \longrightarrow Q) \longleftrightarrow [(\sim Q) \longrightarrow (\sim P)]$ | $R \longrightarrow R$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | T | T | T |
| T | F | F | T | T | T |
| F | T | T | T | T | T |
| F | T | F | T | T | T |
| F | F | T | T | T | T |
| F | F | F | T | T | T |

So when all possible underlying independent component statements are included, we see the truth table columns of these tautologies are indeed the same (all T's!). Similarly all contradictions are equivalent.

To demonstrate how one would prove these, we prove here the first two, (1.40) and (1.41), using a truth table. Notice that all entries for $\mathcal{T}$ are simply T:

| $P$ | $\mathcal{T}$ | $P \vee \mathcal{T}$ | $P \wedge \mathcal{T}$ |
|---|---|---|---|
| T | T | T | T |
| F | T | T | F |

Equivalence (1.40) is demonstrated by the equivalence of the second and third columns, while (1.41) is shown by the equivalence of the first and fourth columns. The others are left as exercises.

These are also worth reflecting upon. Consider the equivalence $P \wedge \mathcal{T} \iff P$. When we use $\wedge$ to connect $P$ to a statement which is always true, then the truth of the compound statement only depends upon the truth of $P$. There are similar explanations for the rest of (1.40)–(1.43).

Some other interesting equivalences involving these are the following:

$$\mathcal{T} \longrightarrow P \iff P \tag{1.44}$$

$$P \longrightarrow \mathcal{F} \iff \sim P. \tag{1.45}$$

We leave the proofs of these for the exercises. These are in fact interesting to interpret. The first says that if a true statement implies $P$, that is the same as in fact having $P$. The second says that if $P$ implies a false statement, that is the same as having $\sim P$, i.e., as having $P$ false. Both types of reasoning are useful in mathematics and other disciplines.

If a statement contains *only* $\mathcal{T}$ or $\mathcal{F}$, then in fact that statement itself must be a tautology ($\mathcal{T}$) or a contradiction ($\mathcal{F}$). This is because there is only one possible combination of truth values. For instance, consider the statement $\mathcal{T} \longrightarrow \mathcal{F}$, which is a contradiction. One proof is in the table:

| $\mathcal{T}$ | $\mathcal{F}$ | $\mathcal{T} \longrightarrow \mathcal{F}$ |
|---|---|---|
| T | F | F |

Since the component statement $\mathcal{T} \longrightarrow \mathcal{F}$ always has truth value F, it is a contradiction. Thus $\mathcal{T} \longrightarrow \mathcal{F} \iff \mathcal{F}$.

## Exercises

Some of these were solved within the section. It is useful to attempt them here again, in the context of the other problems. Unless otherwise specified, all proofs should be via truth tables.

1. Prove (1.18): $\sim(\sim P) \iff P$.

2. Prove (1.32):
$$P \longleftrightarrow Q \iff (\sim P) \longleftrightarrow (\sim Q).$$

3. Prove the logical equivalence of the contrapositive (1.30):
$$P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P).$$

4. Prove (1.29): $P \longrightarrow Q \iff (\sim P) \vee Q$.

5. Show that $P \longrightarrow Q$ and $Q \longrightarrow P$ are not equivalent.

6. Prove De Morgan's Laws (1.19) and (1.20), which are listed again below:
   (a) $\sim(P \vee Q) \iff (\sim P) \wedge (\sim Q)$.
   (b) $\sim(P \wedge Q) \iff (\sim P) \vee (\sim Q)$.

7. Use truth tables to prove the *distributive-type* laws (1.27) and (1.28):
   (a) $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$.
   (b) $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$.

8. Repeat the previous problem but using circuit diagrams.

9. Prove (1.33): $P \longrightarrow (Q \wedge R)$

   $\iff (P \longrightarrow Q) \wedge (P \longrightarrow R)$.

10. Prove (1.34): $P \longrightarrow (Q \vee R)$

    $\iff (P \longrightarrow Q) \vee (P \longrightarrow R)$.

11. Prove (1.21):

    $\sim (P \longrightarrow Q) \iff P \wedge (\sim Q)$.

12. Prove (1.35), which we write below as

    $P \longleftrightarrow Q \iff (P \longrightarrow Q) \wedge (Q \longrightarrow P)$. Note that this justifies the choice of the double-arrow notation $\longleftrightarrow$.

13. Prove (1.22): $\sim (P \longleftrightarrow Q)$

    $\iff (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.

14. Recall the description of **XOR** in footnote 8, page 5.

    (a) Construct a truth table for $P$ **XOR** $Q$.

    (b) Compare to the previous problem. Can you make a conclusion?

    (c) Find an expression for $P$ **XOR** $Q$ using $P$, $Q$, $\sim$, $\wedge$ and $\vee$.

15. Prove that $(P \vee Q) \longrightarrow (P \wedge Q)$ is equivalent to $P \longleftrightarrow Q$. How would you explain in words why this is reasonable? (Perhaps you can think of a colloquial way to verbalize the statement so it will sound equivalent to $P \longleftrightarrow Q$.)

16. Prove (1.31). How would you explain in words why this is reasonable?

17. Prove the following:

    (a) (1.40): $P \vee \mathcal{T} \iff \mathcal{T}$.

    (b) (1.41): $P \wedge \mathcal{T} \iff P$.

    (c) (1.42): $P \vee \mathcal{F} \iff P$.

    (d) (1.43): $P \wedge \mathcal{F} \iff \mathcal{F}$.

18. For each of the following, find a simple, equivalent statement, using truth tables if necessary.

    (a) $\mathcal{T} \vee \mathcal{T}$

    (b) $\mathcal{F} \vee \mathcal{F}$

    (c) $\mathcal{T} \vee \mathcal{F}$

    (d) $\mathcal{T} \wedge \mathcal{T}$

    (e) $\mathcal{F} \wedge \mathcal{F}$

    (f) $\mathcal{T} \wedge \mathcal{F}$

    (g) $\mathcal{T} \longrightarrow \mathcal{T}$

    (h) $\mathcal{F} \longrightarrow \mathcal{F}$

    (i) $\mathcal{T} \longrightarrow \mathcal{F}$

    (j) $\mathcal{F} \longrightarrow \mathcal{T}$

19. Repeat the previous exercise for the following:

    (a) $\mathcal{T} \longrightarrow P$

    (b) $\mathcal{F} \longrightarrow P$

    (c) $P \longrightarrow \mathcal{T}$

    (d) $P \longrightarrow \mathcal{F}$

    (e) $P \longleftrightarrow \mathcal{T}$

    (f) $P \longleftrightarrow \mathcal{F}$

    (g) $P \longleftrightarrow (\sim P)$

    (h) $P \longrightarrow (\sim P)$

20. Prove the associative rules (1.25) and (1.26), page 22.

21. Prove (1.36), page 22.

22. Show $(P \vee Q) \rightarrow R \iff [(P \rightarrow R) \wedge (Q \rightarrow R)]$. Try to explain why this makes sense.

23. Show $(P \wedge Q) \rightarrow R \iff [(P \rightarrow R) \vee (Q \rightarrow R)]$. (This is not so easily explained as is the previous exercise.)

24. There is a notion in logic theory regarding "strong" versus "weak" statements, the stronger ones claiming in a sense more information regarding the underlying statements such as $P, Q$. For instance, $P \wedge Q$ is considered "stronger" than $P \vee Q$, because $P \wedge Q$ tells us more about $P, Q$ (both are declared true) than $P \vee Q$ (at least one is true but both may be). Similarly $P \leftrightarrow Q$ is stronger than $P \rightarrow Q$.

    Each of the following statements may appear "strong" but in fact give little interesting content regarding $P, Q$. Construct a truth table for each and use the truth tables to then explain why they are not terribly "interesting" statements to make about $P, Q$. (Hint: what are these equivalent to?)

    (a) $(P \rightarrow Q) \vee (Q \rightarrow P)$

    (b) $(P \rightarrow Q) \rightarrow (P \leftrightarrow Q)$

## 1.3 Valid Implications and Arguments

Most theorems in this text are in the form of implications, rather than the more rigid equivalences of the last section. Indeed, our theorems are usually of the form "hypothesis implies conclusion." So we have need of an analog to our valid equivalences, namely a notion of valid implications.

### 1.3.1 Valid Implications Defined

Our definition of valid implications is similar to our previous definition of valid equivalences:

**Definition 1.3.1** *Suppose that $R$ and $S$ are compound statements of some independent component statements $P_1, \cdots, P_n$. If $R \longrightarrow S$ is a tautology (always true), then we write*

$$R \implies S, \tag{1.46}$$

*which we then call a* **valid logical implication**.[22]

**Example 1.3.1** *Perhaps the simplest example is the following: $P \implies P$. This seems obvious enough on its face. It can be proved using a truth table (note the vacuous case):*[23]

| $P$ | $P$ | $P \longrightarrow P$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

Thus we see that there are logical implications which are tautologies. A slightly more complicated—and very instructive—example is the following:

**Example 1.3.2** *The following is a valid implication:*

$$(P \wedge Q) \implies P. \tag{1.47}$$

*To prove this, we will use a truth table to show that the following is a tautology:*

$$(P \wedge Q) \longrightarrow P.$$

| $P$ | $Q$ | $P \wedge Q$ | $P$ | $(P \wedge Q) \longrightarrow P$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | F | F | T |
| F | F | F | F | T |

*Notice that three of the four cases have the implication true vacuously.*

The above example is fairly easy to interpret: if $P$ and $Q$ are true, then (of course) $P$ is true. Another intuitive example follows, bascially stating that if we have bi-implication then we have implication.

[22] In this text we will differentiate between implications as statements, such as $P \longrightarrow Q$, which may be true or false, and **valid** implications which are declarations that a particular implication is always true. For example $R \implies S$ means $R \longrightarrow S$ is a tautology. (We similarly differentiated $\longleftrightarrow$ from $\iff$.)

[23] We could also show that $P \longrightarrow P$ is a tautology by way of previously proved results. For instance, with $P \longrightarrow Q \iff (\sim P) \vee Q$ ((1.29), page 22), with the part of $Q$ played by $P$, we have

$$P \longrightarrow P \iff (\sim P) \vee P \iff \mathcal{T},$$

the second statement being in effect our original example of a tautology (Example 1.1.9, page 11).

**Example 1.3.3** *The following is a valid implication:*

$$P \longleftrightarrow Q \implies P \longrightarrow Q. \tag{1.48}$$

*As before, we prove that replacing* $\implies$ *with* $\longrightarrow$ *gives us a tautology.*

| $P$ | $Q$ | $P \longleftrightarrow Q$ | $P \longrightarrow Q$ | $(P \longleftrightarrow Q) \longrightarrow (P \longrightarrow Q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | F | T | T |
| F | F | T | T | T |

*Also as before, we see the importance of the vacuous cases in the final implication. In fact, this is just an application of the previous implication* (1.47)*, if we remember that* $P \longleftrightarrow Q$ *is equivalent to* $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$*:*

$$\underbrace{(P \longrightarrow Q)}_{\text{“}P\text{”}} \wedge \underbrace{(Q \longrightarrow P)}_{\text{“}Q\text{”}} \implies \underbrace{(P \longrightarrow Q)}_{\text{“}P\text{”}}.$$

*The quotes indicate what roles in* (1.47) *are played by the parts of* (1.48).

Another interesting valid implication is given next. (The reader should reflect on its apparent meaning.)

**Example 1.3.4** $(P \longrightarrow Q) \wedge (Q \longrightarrow R) \implies (P \longrightarrow R)$.

*Note that to prove this, we must show that the following statement is a tautology:*

$$[(P \longrightarrow Q) \wedge (Q \longrightarrow R)] \longrightarrow (P \longrightarrow R).$$

| $P$ | $Q$ | $R$ | $P \to Q$ | $Q \to R$ | $(P \to Q) \wedge (Q \to R)$ | $P \to R$ | $[(P \to Q) \wedge (Q \to R)] \longrightarrow (P \to R)$ |
|---|---|---|---|---|---|---|---|
| *T* | *T* | *T* | *T* | *T* | *T* | *T* | *T* |
| *T* | *T* | *F* | *T* | *F* | *F* | *F* | *T* |
| *T* | *F* | *T* | *F* | *T* | *F* | *T* | *T* |
| *T* | *F* | *F* | *F* | *T* | *F* | *F* | *T* |
| *F* | *T* | *T* | *T* | *T* | *T* | *T* | *T* |
| *F* | *T* | *F* | *T* | *F* | *F* | *T* | *T* |
| *F* | *F* | *T* | *T* | *T* | *T* | *T* | *T* |
| *F* | *F* | *F* | *T* | *T* | *T* | *T* | *T* |

At this stage in the development it is perhaps best to check such things using truth tables, however unwieldy they can be. Of course with practice comes intuition, informed memory and many shortcuts, but for now we will use this brute force method of truth tables to determine if an implication is valid.

For a simple algebraic perspective of the difference between a valid implication and a valid equivalence, consider the following list of algebraic facts:

$$\begin{aligned} x = 5 &\implies x^2 = 25, \\ x = -5 &\implies x^2 = 25, \\ x^2 = 25 &\iff (x = 5) \vee (x = -5). \end{aligned}$$

Knowing $x = 5$ is not *equivalent* to knowing $x^2 = 25$. That is because there is an alternative explanation for $x^2 = 25$, namely that perhaps $x = -5$. But it is true that knowing $x = 5$ *implies* knowing—at least in principle—that $x^2 = 25$ (just as knowing $x = -5$ implies knowing $x^2 = 25$). If an equivalence is desired, a valid one is that knowing $x^2 = 25$ *is* equivalent to knowing that $x$ must be either 5 or $-5$.

Later in the text we will briefly focus on algebra in earnest, bringing our symbolic logic to bear on that topic. In algebra (and in calculus) it is often important to know when we have an equivalence and when we have only an implication. For some algebraic problems, the implication often means we need to check our answer while the equivalence means we do not. For an example of this phenomenon, consider

$$\begin{aligned}\sqrt{x+2} = x \implies x+2 = x^2 \iff 0 = x^2 - x - 2 \iff 0 = (x-2)(x+1)\\ \iff (x-2=0) \vee (x-1=0) \iff (x=2) \vee (x=-1).\end{aligned}$$

We lost the equivalence at the first step, and so we can only conclude from the logic that

$$\sqrt{x+2} = x \implies (x=2) \vee (x=-1).$$

All this tells us is that *if* there is a a number $x$ so that $\sqrt{x+2} = x$, then the number must be either $x = 2$ or $x = -1$ (or perhaps both work; recall that we always interpret *or* inclusively). When we check $x = 2$ in the original equation, we get $\sqrt{4} = 2$, which is true. However, $x = -1$ gives $\sqrt{1} = -1$, which is not true. Since we have now solved the original equation we can say that[24]

$$\sqrt{x+2} = x \iff x = 2.$$

For another example, consider how one can solve a linear equation:

$$2x + 1 = 3 \iff 2x = 2 \iff x = 1.$$

Here we subtracted 1 from both sides, and then divided by 2, neither of which break the logical equivalence. We do not have to check the answer (unless we believe our arithmetic or reasoning may be faulty). In Chapter 2 we will delve more deeply into algebra.

### 1.3.2 Partial List of Valid Implications

Table 1.4, page 32 lists some basic valid equivalences and implications. All can be proved using truth tables. However, it is important to learn to recognize validity without always resorting to truth tables. Each can be viewed in light of English examples. Still, it is the rigorous *mathematical* framework which gives us the precise rules for rewriting and analyzing statements.[25]

It is useful to see why (1.49)–(1.58) are *not* equivalences.[26] For instance a little reflection should make clear that[27] $P \not\Longleftarrow P \wedge Q$ (unless there is some underlying relationship between $P$ and $Q$ which is not stated), and so we cannot replace $\implies$ with $\iff$ in (1.49). Similarly, in (1.52) having $P \longrightarrow R$ in itself says nothing about $Q$, so there is no reason to believe $(P \longrightarrow Q) \wedge (Q \longrightarrow R)$ is implied by $P \longrightarrow R$.

Implicit in the above discussion is the fact that having $\iff$ is the same as simultaneously having both $\implies$ and $\Longleftarrow$. Put another way, $R \iff S$ is the same as collectively having

[24]Later, in Chapter 2 we will define $\sqrt{z}$ to be only the nonnegative square root of $z$, assuming $z \geq 0$ lest $\sqrt{z}$ be undefined, at least as a *real* number.

[25]Similarly we use the rules of algebra to rewrite and analyze equations in hopes of solving for the variables.

[26]The exceptions in the table are (1.60) and (1.61), which are valid if we replace $\implies$ with $\iff$, as was discussed in the previous section. See (1.45), page 27 and the exercises of that section.

[27]Note that it is common to negate a statement by including a "slash" through the main symbol, as in $\sim(x=3) \iff x \neq 3$. What we mean by $R \not\Longrightarrow S$ is that it is not true that $R \longrightarrow S$ is a tautology.

$$P \wedge Q \implies P \tag{1.49}$$
$$P \implies P \vee Q \tag{1.50}$$
$$P \longleftrightarrow Q \implies P \longrightarrow Q \tag{1.51}$$
$$(P \longrightarrow Q) \wedge (Q \longrightarrow R) \implies P \longrightarrow R \tag{1.52}$$
$$(P \longleftrightarrow Q) \wedge (Q \longrightarrow R) \implies (P \longrightarrow R) \tag{1.53}$$
$$(P \longrightarrow Q) \wedge (Q \longleftrightarrow R) \implies (P \longrightarrow R) \tag{1.54}$$
$$(P \longleftrightarrow Q) \wedge (Q \longleftrightarrow R) \implies (P \longleftrightarrow R) \tag{1.55}$$
$$(P \longrightarrow Q) \wedge P \implies Q \tag{1.56}$$
$$(P \longrightarrow Q) \wedge (\sim Q) \implies \sim P \tag{1.57}$$
$$(P \vee Q) \wedge (\sim Q) \implies P \tag{1.58}$$
$$P \longrightarrow (\sim P) \implies \sim P \tag{1.59}$$
$$P \longrightarrow \mathcal{F} \implies \sim P \tag{1.60}$$
$$\mathcal{T} \longrightarrow P \implies P \tag{1.61}$$

**Table 1.4:** Table of Valid Logical Implications. If we replace $\implies$ with $\longrightarrow$ in each of the above (perhaps enclosing each side in brackets $[\cdots]$), we would have tautologies.

both $R \implies S$ and $S \implies R$.[28] In fact it is important to note that all of the valid logical equivalences, for instance in Table 1.3, page 22 can be also considered to be combinations of two valid implications, one with $\implies$, and the other with $\impliedby$, replacing $\iff$. We do not list them all here, but rather list the most commonly used implications which are not equivalences, except for the last three in the table.

### 1.3.3 Fallacies and Valid Arguments[29]

> *The name* **fallacy** *is usually reserved for* typical *faults in arguments that we nevertheless find persuasive. Studying them is therefore a good defense against deception.*
> —Peter Suber, Department of Philosophy, Earlham College, Richmond, Indiana, 1996.

Here we look at some classical argument styles, some of which are valid, and some of which are invalid and therefore called *fallacies* (whether or not they may seem persuasive at first glance). The valid styles will mostly mirror the valid logical implications of Table 1.4.

A common method for diagramming simple arguments is to have a horizontal line separating the *premises*[30] from the *conclusions*. Usually we will have multiple premises and a single conclusion. For style considerations, the conclusion is often announced with the symbol $\therefore$ which is read "therefore."[31]

In this subsection we look at several of these arguments, both valid and fallacious. Many are classical, with classical names. We will see how to analyze arguments for validity. In all cases here, it will amount to determining if a related implication is valid.

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[28] Note that $R \impliedby S$ would be interpreted as $S \implies R$. We will not make extensive use of "$\impliedby$."

[29] Valid argument forms are also called *rules of inference*.

[30] Premises are also called *hypotheses*. The singular forms are premise and hypothesis.

[31] In fact most texts use either the horizontal line or the symbol $\therefore$ but not both. We use both to emphasize where the hypotheses end and the conclusion begins.

**Example 1.3.5** *Our first example we consider is the argument form which is classically known as* **modus ponens**, *or* **law of detachment**. *It is outlined as follows:*[32]

$$\begin{array}{c} P \longrightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

*The idea is that if we* **assume** $P \longrightarrow Q$ *and* $P$ *are true, then we must* **conclude** *that* $Q$ *is also true. This is ultimately an implication. The key is that checking to see if this is valid is the same as checking to see if*

$$(P \longrightarrow Q) \wedge P \implies Q,$$

*i.e., that* $[(P \longrightarrow Q) \wedge P] \longrightarrow Q$ *is a tautology. We know this to be the case already, as this is just* (1.56), *though we should* **prove** *this by producing the relevant truth table to show that* $[(P \longrightarrow Q) \wedge P] \longrightarrow Q$ *is indeed a tautology, i.e., has truth value* T *for all cases of truth values of* $P, Q$*:*

| $P$ | $Q$ | $P \rightarrow Q$ | $(P \rightarrow Q) \wedge P$ | $Q$ | $[(P \rightarrow Q) \wedge P] \longrightarrow Q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | F | T |
| F | T | T | F | T | T |
| F | F | T | F | F | T |

Thus to test the validity of an argument is to test whether or not the argument, written as an implication, is a tautology. This gives us a powerful, *computational* tool to analyze the classical argument styles. It also connects some of our symbolic logic to this style of diagramming arguments, so the intuition of these two flavors of logic can illuminate each other.

To repeat and emphasize the criterion for validity we list the following definition:

**Definition 1.3.2** *A* **valid argument** *is one which, when diagrammed as an implication, represents a tautology. In other words, if the premises are* $\mathcal{P}_1, \mathcal{P}_2 \cdots, \mathcal{P}_m$ *and the conclusion is* $\mathcal{Q}$ *(where* $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_m$ *and* $\mathcal{Q}$ *are compound statements based upon some underlying independent statements* $P_1, \cdots, P_n$*), then the argument is valid if and only if*

$$\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \cdots \wedge \mathcal{P}_m \Longrightarrow \mathcal{Q},$$

*i.e., if and only if* $[\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \cdots \wedge \mathcal{P}_m] \longrightarrow \mathcal{Q}$ *is a tautology. If not, then the argument is a called a* **fallacy**.

Note that the validity of any argument does not depend upon the truth or falsity of the conclusion. Indeed the *modus ponens* argument in Example 1.3.5 is perfectly valid, regardless of whether or not $Q$ is true. That is because we do not know—or even ask for purposes of discovering if the logic is valid—whether or not the premises are true. What we do know is that, *if the premises are true, then so is the conclusion.* In other words, the statement $[(P \rightarrow Q) \wedge P] \longrightarrow Q$ is always true. (If one or more of the premises are false, the implication is true vacuously.)

[32] *Modus ponens* is short for *modus ponendo ponens*, which is Latin for "the way that affirms by affirming." It is important enough that it has been extensively studied through the ages, and thus has many names, another being "affirmation argument." As for "law of detachment," it is pointed out in J.E. Rubin's *Mathematical Logic: Applications and Theory* (Saunders, 1990), that the idea is that we can validly "detach" the consequent $Q$ of the conditional $P \longrightarrow Q$ when we also assume the antecedent $P$.

One example often used to shed light on the law of detachment above, and other argument styles as well, uses the following choices for $P$ and $Q$.

$$\begin{aligned} P &: \text{It rained} \\ Q &: \text{The ground is wet} \end{aligned}$$

The argument above could then be diagrammed again but using the words represented by $P, Q$:

$$\begin{array}{c} \text{If it rained, then the ground is wet.} \\ \text{It rained.} \\ \hline \therefore \text{The ground is wet.} \end{array}$$

This is a perfectly valid argument, meaning that if we accept the premises we must accept the conclusion. In other words, the logic is flawless. That said, one need not necessarily accept the conclusion just because the argument is valid, since one can always debate the truthfulness of the premises. Again the key is that the logic here is valid, even if the premises may be faulty.[33]

Next we look at an example of an invalid argument, i.e., a fallacy. The following is called the **fallacy of the converse**:[34]

**Example 1.3.6** *Show that the following argument is a fallacy:*

$$\begin{array}{c} P \longrightarrow Q \\ Q \\ \hline \therefore P \text{ (Invalid)} \end{array}$$

*As before, we analyze the corresponding implication, in this case* $[(P \to Q) \wedge Q] \longrightarrow P$*, with a truth table:*

| $P$ | $Q$ | $P \to Q$ | $(P \to Q) \wedge Q$ | $P$ | $[(P \to Q) \wedge Q] \longrightarrow P$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | F | T | F | F | T |

It is always useful to review an invalid argument to see which conditions were problematic. In the third row of our truth table, $P \to Q$ is vacuously true, and $Q$ is true so the premises hold true, but the conclusion $P$ is false (which was why $P \to Q$ was vacuously true!). From a more common-sense standpoint, while $P \to Q$ is assumed, $P$ may not be the only condition which forces $Q$ to be true. (If it were, we would instead have $P \longleftrightarrow Q$.) Consider again our previous choices for $P$ and $Q$:

$$\begin{array}{c} \text{If it rained, then the ground is wet.} \\ \text{The ground is wet.} \\ \hline \therefore \text{It rained. (Invalid)} \end{array}$$

[33]In mature philosophical discussions, the logic is rarely in question because the valid models of argument are well known. When a conclusion seems unacceptable or just questionable, it is usually the premises which then come under scrutiny.

[34]The *converse* of an implication $R \longrightarrow S$ is the statement $S \longrightarrow R$ (or $R \longleftarrow S$ if we want to preserve the order). An implication and its converse are not logically equivalent, as a quick check of their truth tables would reveal. However, it is a common mistake to forget which direction an implication follows, or to just be careless and mistake an implication for a bi-implication. The "fallacy of the converse" refers to a state of mind where one mistakenly believes the converse true, based upon the assumption that the original implication is true. (Note that if we mistakenly replace $P \longrightarrow Q$ with $Q \longrightarrow P$ in Example 1.3.6, the new argument would be valid. In fact it is *modus ponens.*)

Even if the premises are correct in Example 1.3.6, the ground being wet does not guarantee that it rained. Perhaps it is wet from dew, or a sprinkler, or flooding from some other source. Here one can accept the premises, but the conclusion given above is not valid.

There is a subtle—perhaps difficult—general point in this subsection which bears repeating: the truth table associated with an argument reflects the validity or invalidity of the logic of the **argument** (i.e., the validity of the corresponding implication), regardless of the truthfulness of the premises. Indeed, note how the truth table for the valid form *modus ponens* of Example 1.3.5 (page 33) contains cases where the premises, $P \longrightarrow Q$ and $P$, can have truth value F as well as T.[35]

For completeness, we mention that some use the adjective **sound** to describe an argument which is not only valid, but whose premises (and therefore conclusions) are in fact true. Of course in reality those are usually the arguments which we seek, but (arguably) one must first understand validity before probing the soundness of arguments, and so for this text, we are mostly interested in abstract, valid arguments, and worry about soundness only in context.

The next example is also very common. It is a valid form of argument often known by its Latin name **modus tollens**.[36]

**Example 1.3.7** *Analyze the following* **(modus tollens)** *argument.*

$$\begin{array}{c} P \longrightarrow Q \\ \underline{\sim Q \quad} \\ \therefore \sim P \end{array}$$

*As before, we analyze the following associated implication (which we leave as a single-arrow implication until we establish it is a tautology):*

$$[(P \to Q) \wedge (\sim Q)] \longrightarrow (\sim P).$$

| $P$ | $Q$ | $P \to Q$ | $\sim Q$ | $(P \to Q) \wedge (\sim Q)$ | $\sim P$ | $[(P \to Q) \wedge (\sim Q)] \longrightarrow (\sim P)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | T | T | T |

*That the final column is all* T*'s thus establishes its validity. In fact, we see that the argument above is just a re-diagrammed version of* (1.57), page 32*.*

A short chapter could be written just on the insights which can be found studying the above *modus tollens* argument. For instance, for a couple of reasons one could make the case that *modus tollens* and *modus ponens* are the same type of argument. We will see one of these reasons momentarily, but first we will look at *modus tollens* by itself. Inserting our previous $P$ and $Q$ into this form, we would have

[35] Another, perhaps more subtle point here is the extensive role of the vacuous cases in the underlying implication of an argument. If the premises are not true, then the conclusion can not contradict them. This gives rise to some strange forms of argument indeed, for the premises can be self-contradictory and therefore, taken as a group of statements joined by $\wedge$, can be equivalent to $\mathcal{F}$. (Recall $\mathcal{F} \to P$ is a tautology.) However, the only practical uses of arguments come when we know the premises to be true, or we think they are false and demonstrate it by showing the valid conclusions they imply are demonstrably false. The latter use is often called *proof by contradiction* or *indirect proof*, but there are many structures which use the same idea. *Modus tollens* (Example 1.3.7) is one permutation of the idea behind indirect proof.

[36] Short for *modus tollendo tollens,* Latin for "the way that denies by denying." It is also called "denying the consequent," which contrasts it to "affirming the consequent," another name for the fallacy of the converse, Example 1.3.6, page 34. As the reader can deduce, most of these common arguments—valid or not—have many names, inspired by different contexts and considerations. Computationally they are simple enough, once seen as implications to be analyzed and found to be tautologies (in cases of valid implications) or non-tautologies (in cases of fallacies).

$$\begin{array}{c} \text{If it rained, then the ground is wet.} \\ \text{The ground is not wet.} \\ \hline \therefore \text{ It did not rain.} \end{array}$$

The validity of the above argument should be intuitive. A common way of explaining it is that it must not have rained, because (first premise) if it had rained the ground would be wet, and (second premise) it is not wet. Of course that explanation is probably no simpler than just reading the argument as it stands.

Another way to look at it is to recall the equivalence of the implication to the contrapositive ((1.30), page 22 and elsewhere):

$$P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P).$$

Thus we can replace in the *modus tollens* argument the first premise, $P \to Q$, with its (equivalent) contrapositive:

$$\begin{array}{c} (\sim Q) \longrightarrow (\sim P) \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

This is valid by *modus ponens* (Example 1.3.5, page 33), with the part of $P$ there played by $\sim Q$ here, and the part of $Q$ by $\sim P$. In fact the next valid argument form further unifies *modus ponens* and *modus tollens*, as will be explained below, though this next form is interesting in its own right.

**Example 1.3.8** *Consider the following form of argument, called* disjunctive syllogism, *which is valid.*[37]

$$\begin{array}{c} P \vee Q \\ \sim P \\ \hline \therefore Q \end{array}$$

*The proof of this is left as an exercise. To prove this one needs to show*

$$(P \vee Q) \wedge (\sim P) \Longrightarrow Q,$$

*that is, to show* $[(P \vee Q) \wedge (\sim P)] \longrightarrow Q$ *to be a tautology. Of course the idea of this argument style is that when we assume "$P$ or $Q$" to be true, and then further assume $P$ is false (by assuming "$\sim P$" is true), we are forced to conclude $Q$ must be true. Note that this is just a re-diagrammed version of* (1.58), page 32 *except with $P$ and $Q$ exchanging roles. For an example, we will use a different pair of statements $P$ and $Q$:*

$$P : \textit{I will eat pizza}$$

$$Q : \textit{I will eat spaghetti}$$

*The argument above becomes (after minor colloquial adjustment):*

$$\begin{array}{c} \textit{I will eat pizza or spaghetti.} \\ \textit{I will not eat pizza.} \\ \hline \therefore \textit{ I will eat spaghetti.} \end{array}$$

To see how this unifies both *modus ponens* and *modus tollens* as two manifestations of the same principle, recall the following (easily proved by a truth table):

$$P \longrightarrow Q \iff (\sim P) \vee Q.$$

This appeared as (1.29), page 22, for instance. Thus the *modus ponens* and *modus tollens* become, respectively,

[37] A lay person might call this a form of "process of elmination."

$$\begin{array}{c} (\sim P) \vee Q \\ \sim(\sim P) \\ \hline \therefore Q, \end{array} \qquad \text{and} \qquad \begin{array}{c} (\sim P) \vee Q \\ \sim Q \\ \hline \therefore \sim P. \end{array}$$

Compare these to the original forms, respectively, to see they are the same:

$$\begin{array}{c} P \longrightarrow Q \\ P \\ \hline \therefore Q, \end{array} \qquad \text{and} \qquad \begin{array}{c} P \longrightarrow Q \\ \sim Q \\ \hline \therefore \sim P. \end{array}$$

For another fallacy, consider the **fallacy of the inverse**:[38]

**Example 1.3.9** *Show that the following statement is a fallacy.*

$$\begin{array}{c} P \longrightarrow Q \\ \sim P \\ \hline \therefore \sim Q \text{ (Invalid)} \end{array}$$

*Solution: We check to see if the statement*

$$[(P \to Q) \wedge (\sim P)] \longrightarrow (\sim Q)$$

*is a tautology (in which case we could replace the major operation $\longrightarrow$ with $\Longrightarrow$).*

| $P$ | $Q$ | $P \to Q$ | $\sim P$ | $(P \to Q) \wedge (\sim P)$ | $\sim Q$ | $[(P \to Q) \wedge (\sim P)] \longrightarrow (\sim Q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T |
| T | F | F | F | F | T | T |
| F | T | T | T | T | F | F |
| F | F | T | T | T | T | T |

*We note that the argument—as an implication—is false in the case $P$ is false and $Q$ is true. In fact, in that case we have both premises of the argument ($P \to Q$ vacuously, and $\sim P$ obviously), but not the conclusion. Let us return to our rain and wet grass statements from before:*

$$\begin{array}{c} \textit{If it rained then the grass is wet.} \\ \textit{It did not rain.} \\ \hline \therefore \textit{The grass is not wet. (Invalid)} \end{array}$$

*For another example, consider the following obviously fallacious argument:*

$$\begin{array}{c} \textit{If you drink the hemlock then you will die.} \\ \textit{You do not drink the hemlock.} \\ \hline \therefore \textit{You will not die. (Invalid)} \end{array}$$

*There are, of course, other reasons why you might die (or why the grass might be wet). The case we see in the truth table example which ruins the bid for the corresponding implication to be a tautology is that case in which you do not drink the hemlock, and still die, contradicting the conclusion but not the hypotheses.*

[38] The **inverse** of an implication $R \longrightarrow S$ is the statement $(\sim R) \longrightarrow (\sim S)$. It is **not** equivalent to the original implication. In fact, it is equivalent to the converse (see Footnote 34, page 34), the proof of which is left to the exercises. A course on logic would emphasize these two statements which are related to the implication. However, they are a source of some confusion so we do not elaborate extensively here. It is much more important to realize that $R \longrightarrow S$ is equivalent to its contrapositive $(\sim S) \longrightarrow (\sim R)$, and **not** equivalent to these other two related implications, namely the converse $S \longrightarrow R$ and inverse $(\sim R) \longrightarrow (\sim S)$. These facts should become more self-evident as the material is studied and utilized.

There are many other forms of argument, both valid and invalid. A rudimentary strategy for detecting if an argument is valid or a fallacy is the same: to look at the corresponding implication and see if it is a tautology. If so, the argument is valid and if not, the argument is a fallacy. In the next subsection we introduce a slightly more sophisticated method, in which we use previously established valid implications and equivalences to make shorter work of some complicated arguments, but for now we will continue to use the truth table test of the validity of the underlying implication.

Our next example was proved earlier in the form of a valid implication. As such it was the subject of Example 1.3.4, page 30, and was listed as (1.52), page 32. We will not rework the truth table here.

**Example 1.3.10** *The following is a valid form of argument:*

$$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow R \\ \hline \therefore P \longrightarrow R \end{array}$$

*For an example of an application, consider the following assignments for $P, Q$ and $R$:*

$$\begin{aligned} &P : \textit{I am paid} \\ &Q : \textit{I will buy you a present} \\ &R : \textit{You will be happy} \end{aligned}$$

*The argument then becomes the following, the validity of which is reasonably clear:*

$$\begin{array}{c} \textit{If I am paid, then I will buy you a present.} \\ \textit{If I will buy you a present, then you will be happy.} \\ \hline \therefore \textit{If I am paid, then you will be happy.} \end{array}$$

Next we look at an example which contains three underlying component statements, and three premises. Before doing so, we point out that we can compute the truth tables for $P \wedge Q \wedge R$ and $P \vee Q \vee R$ relatively quickly; the former is true whenever all three are true (and false if at least one is false), and the latter is true if any of the three are true (and false only if all three are false). The reason we can do this is that there is no ambiguity in computing, for instance, $P \wedge (Q \wedge R)$ or $(P \wedge Q) \wedge R$, as these are known to be equivalent (see page 22). Similarly for $\vee$.[39] To be clearer, we note the truth tables.

| $P$ | $Q$ | $R$ | $P \wedge Q \wedge R$ | $P \vee Q \vee R$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | F | T |
| T | F | T | F | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | T | F | F | T |
| F | F | T | F | T |
| F | F | F | F | F |

[39] This is similar to the arithmetic rules that $A+B+C = A+(B+C) = (A+B)+C$. The first expression is not at first defined *per se*, but because the second and third are the same we allow for the first. Similarly $A \cdot B \cdot C = A \cdot (B \cdot C) = (A \cdot B) \cdot C$. However, this does not extend to all operations, such as subtraction: Usually $A-(B-C) \neq (A-B)-C$, so when we write $A-B-C$ we have to choose one, and in fact we choose the latter. Similarly, we have to be careful with other logical operations besides $\wedge$ and $\vee$. For instance, $P \longrightarrow Q \longrightarrow R$ would probably be interpreted $(P \longrightarrow Q) \longrightarrow R$, but it would depend upon the author.

With this observation, we can more easily analyze arguments with more than two premises.

**Example 1.3.11** *Consider the argument*

$$\begin{array}{c} P \longrightarrow (Q \vee R) \\ P \\ \sim R \\ \hline \therefore Q \end{array}$$

*We need to see if the following conditional—which we dub ARG (for "argument") for space considerations—is a tautology:*

$$ARG: \qquad \{[P \longrightarrow (Q \vee R)] \wedge P \wedge (\sim R)\} \longrightarrow Q.$$

| $P$ | $Q$ | $R$ | $Q \vee R$ | $P \to (Q \vee R)$ | $\sim R$ | $(P \to (Q \vee R)) \wedge P \wedge (\sim R)$ | $Q$ | $ARG$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |

*Note that "ARG" connects the two immediately preceding columns with the logical operation $\longrightarrow$. Also notice that most of the cases are true vacuously (as often happens when we have more premises to be met, connected by $\wedge$), and eventually we see that the argument is valid.*

Reading more examples, and perhaps some trial and error, one's intuition for what is valid and what is not should develop. With English examples one might be able to see that the above argument style is reasonable. For instance, consider the following.

$P$ : I will eat pizza
$Q$ : I will drink soda
$R$ : I will drink beer

Then the premises become

$P \longrightarrow (Q \vee R)$ : If I eat pizza, then I will drink soda or beer.
$P$ : I will eat pizza.
$\sim R$ : I will not drink beer.

It is reasonable to believe that, after declaring that if I eat pizza then I will drink soda or beer, and that I indeed will eat pizza, but not drink the beer, then I must drink the soda.[40]

### 1.3.4 Analyzing Arguments Without Truth Tables

In fact there is another approach for analyzing complicated arguments such as the one above. The strategy is manipulative, and there are two tools we can make use of. The first is the somewhat obvious fact that we can always replace one of the hypotheses with a logically equivalent

[40] Or else I was lying when I recited my premises. The argument, anyhow, is valid.

statement. That is because the truth table column entries are what matter in our computations there. But there is another strategy which is not quite so obvious. It relies upon the following logical equivalence, which is left as an exercise:

$$\mathcal{P} \longrightarrow \mathcal{Q} \iff \mathcal{P} \longleftrightarrow (\mathcal{P} \wedge \mathcal{Q}). \tag{1.62}$$

For our purposes, it means that if our premises $\mathcal{P}$ imply $\mathcal{Q}$, then we can replace them with $\mathcal{P} \wedge \mathcal{Q}$, in effect *attaching* $Q$ to the list of hypotheses. Note also this is true if $\mathcal{P}$ is instead just a single hypothesis in the list of hypotheses, which are joined by "ands" $\wedge$, which is commutative and associative so $\mathcal{Q}$ can be attached anywhere in the list as another hypothesis.

If one or more of the hypotheses taken collectively validly imply a statement $\mathcal{Q}$, then $\mathcal{Q}$ can be attached to the list of hypotheses, again due to the commutative and associative nature of $\wedge$.

**Example 1.3.12** *Let us re-examine the previous example, but this time we append some intermediate conclusions.*

$$\begin{array}{c} \left.\begin{array}{c} \left.\begin{array}{c} P \longrightarrow (Q \vee R) \\ P \end{array}\right\} \therefore Q \vee R \\ \sim R \end{array}\right\} \therefore Q \\ \hline \therefore Q \end{array}$$

*The first conclusion $Q \vee R$ is implied by the first two premises by a simple* modus ponens, *and the second came from the third original hypothesis $\sim R$ and our newly attached hypothesis $(Q \vee R)$ by the way of the disjunctive syllogism (*Example 1.3.8, page 36 *and* (1.57), page 32*). It requires more creativity than a truth table verification, but is clearly less tedious.*

**Example 1.3.13** *Consider the following argument style:*

$$\begin{array}{c} (P \vee Q) \longrightarrow R \\ \sim R \\ \hline \therefore \sim P. \end{array}$$

*Here again we can work backwards. (A reasonable alternative style would skip some steps.)*

$$\begin{array}{c} \left.\begin{array}{c} (P \vee Q) \longrightarrow R \\ \sim R \end{array}\right\} \therefore \sim (P \vee Q)\Big\} \therefore (\sim P) \wedge (\sim Q)\Big\} \therefore \sim P \\ \hline \therefore \sim P. \end{array}$$

Note that in this latest example we could have appended our original hypotheses to include $(\sim P) \wedge (\sim Q)$, which would be the same as appending $\sim P$ and $\sim Q$ separately.

This method is often quite useful for verifying validity, but it is not, by itself, a way to detect a fallacy. However, since common fallacies rest upon the fallacy of the inverse (page 37) or the fallacy of the converse (page 34), we can sometimes detect when an argument tempts us to agree with the conclusion because of such invalid, but common, reasoning.

**Example 1.3.14** *Consider the following argument.*

$$\begin{array}{c} (P \vee Q) \longrightarrow R \\ \sim Q \\ R \\ \hline \therefore P. \text{ (Invalid!)} \end{array}$$

*What is tempting (but invalid) to do with this argument, is to reason that the first and third hypotheses imply $P \vee Q$, and with the second reading $\sim Q$, we would then (seemingly validly)* *conclude $P$. However, we can not concluded $P$ from our premises. Indeed, $R$ does not imply anything about $P$ and $Q$, and if nothing else, a truth table will prove this style invalid. An errant diagram might look like the following:*

$$\left.\begin{array}{c}\boxed{(P\vee Q)\longrightarrow R}\\ \sim Q\\ \boxed{R}\\ \hline \therefore P\end{array}\right\}\therefore (P\vee Q)\qquad \text{(INVALID!)}\qquad \left.\sim Q\right\}\therefore P$$

*An example in English might help to see that indeed this is invalid. If we define $P$, $Q$ and $R$ as below, it seems pretty clear this style is a fallacy.*

$$\begin{aligned} P &: \textit{I shot you.}\\ Q &: \textit{I stabbed you.}\\ R &: \textit{You died.}\end{aligned}$$

*This partuclar argument then reads:*

$$\begin{array}{c}\textit{If I shot you or stabbed you, then you died.}\\ \textit{I did not stab you.}\\ \textit{You died.}\\ \hline \therefore \textit{ I shot you.}\ \text{(Invalid!)}\end{array}$$

*Clearly this is fallacious reasoning. The "alternative explanation" for the consequent, i.e., $\mathcal{Q}$ where $\mathcal{P}\longrightarrow\mathcal{Q}$, should always be looming when we look at implications and try to read them backwards. (In this case there is nothing in the premises that do not allow for other causes of your dying.)*

From the above discussion, one might conclude that we can often intuitively detect the likelihood of an argument being invalid, but unless we can rewrite it as a known fallacy—or clearly see a case where the premises hold true and the conclusion does not[41]—we would need to go back to our primitive but absolutely reliable method of constructing the truth table for the underlying implication to see if we have a tautology. If we do not have a tautolgy then the argument is a fallacy; if we do, then the argument is valid.[42]

Still, this new method of proving validity for arguments can be very useful, especial in lieu of long truth tables, but of course it necessarily rests upon the styles we previously proved to be valid, or the valid equivalences or implications from before. (It also requires a bit of creativity in

[41]Some texts have the reader check the validity of an argument by checking only the cases in which **all** premises are true, to see if the conclusions are also true for those cases. Indeed it is only the cases in which the premises are true and the conclusion is false which invalidate an argument style. However, that kind of analysis de-emphasizes the connection between valid argument styles and valid implications, and the role of tautologies and so we prefer to include these ideas, at the cost of looking at every case of truth values for the underlying statements $P, Q$, and so on when analyzing an argument for validity.

[42]By now the reader, having encountered abstract and applied implications on many abstract levels, should be aware of the reason to write both sides of semicolon in the sentence above, namely, "If we do not have a tautolgy then the argument is a fallacy; if we do, then the argument is valid." To spell this out better, consider

$$\begin{aligned} P &:\text{we have a tautology (in the form of the argument as an implication)}\\ Q &:\text{the argument is valid}\end{aligned}$$

Then the sentence in quotes reads $[(\sim P)\rightarrow(\sim Q)]\wedge[P\rightarrow Q]$, which is equivalent to $[Q\rightarrow P]\wedge[P\rightarrow Q]$, i.e., $P\leftrightarrow Q$. If we wrote only the first part, $(\sim P)\rightarrow(\sim Q)$, that alone would not declare the second part $P\rightarrow Q$, though many casual readers would assume that it would (in words if not symbols).

using them.) The more known valid styles, equivalences and implications one has available, the larger the number of argument styles which can be proven without resorting to truth tables. Of course it was the truth tables which allowed us to prove the preliminary equivalences, implications and argument styles to be valid, so ultimately all these things rest upon the truth tables. They are the one device at our disposal that allowed logic to be rendered computational (and at a very fundamental level), rather than just intuitive.

## Exercises

Some of these were proved previously in the text. The reader should attempt to prove these first for himself/herself without referring back to the text for the proofs.

1. Prove $P \longrightarrow Q \iff P \longleftrightarrow (P \wedge Q)$. This is essentially (1.62), page 40.

2. Consider the statement $P \longrightarrow (\sim P)$.

   (a) Use a truth table to prove the validity of $P \longrightarrow (\sim P) \implies (\sim P)$. Is this reasonable?

   (b) If possible without truth tables, and using a short string of known equivalences, show that in fact

   $$P \longrightarrow (\sim P) \iff (\sim P).$$

   (See (1.29), page 22 and (1.17), page 22.)

3. Use truth tables to show (1.49) and (1.50):

   $$P \wedge Q \implies P,$$
   $$P \implies P \vee Q.$$[43]

4. Prove the following without using truth tables.

   $$P \implies (P \vee Q)$$

   by proving that $P \longrightarrow (P \vee Q)$ is a tautology. (Again, see (1.29), page 22.)

5. Prove the following two valid implications:

   (a) (1.56): $(P \longrightarrow Q) \wedge P \implies Q$.

   (b) (1.57): $(P \longrightarrow Q) \wedge (\sim Q) \implies \sim P$.

6. Prove the following using truth tables. Note that there are $2^4 = 16$ different combinations of truth values for $P$, $Q$, $R$ and $S$.

   $$(P \longrightarrow R) \wedge (Q \longrightarrow S) \implies (P \wedge Q) \longrightarrow (R \wedge S), \quad (1.63)$$

   $$(P \longrightarrow R) \wedge (Q \longrightarrow S) \implies (P \vee Q) \longrightarrow (R \vee S) \quad (1.64)$$

   These are useful because we often have a string of statements $A_1, \cdots, A_n$ connected entirely by $\wedge$ or entirely by $\vee$, and wish to replace them with $B_1, \cdots, B_n$ (or just some of them), where $A_1 \implies B_1$, $A_2 \implies B_2$, $\cdots$, $A_n \implies B_n$.[44] With (1.63) and (1.64) we can generalize and write

   $$A_1 \wedge A_2 \wedge \cdots \wedge A_n \implies B_1 \wedge B_2 \wedge \cdots \wedge B_n, \quad (1.65)$$

   $$A_1 \vee A_2 \vee \cdots \vee A_n \implies B_1 \vee B_2 \vee \cdots \vee B_n. \quad (1.66)$$

   Valid implications (1.63) and (1.64) would be quite laborious to prove without truth tables.

[43] Some would characterize $P \wedge Q \implies P \implies P \vee Q$ to be a progression from the strongest statement, $P \wedge Q$, to the weakest, $P \vee Q$ of the three. A similar form would be $P \wedge Q \implies P \implies P \vee R$.

[44] Here the parts of $P \longrightarrow R$ and $Q \longrightarrow S$ are played by the $A_i \implies B_i$, but the idea is the same. We could instead attach $\mathcal{T} \iff (A_1 \longrightarrow B_1) \wedge \cdots \wedge (A_n \longrightarrow B_n)$ to the left-hand sides of (1.65) and (1.66) with the wedge operation, since $\mathcal{T} \wedge U \iff U$.

7. For each of the following, decide if you believe it is a valid argument or a fallacy. Then check by constructing the corresponding truth table.

(a) $$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow P \\ \hline \therefore P \longleftrightarrow Q \end{array}$$

(b) $$\begin{array}{c} P \longleftrightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

(c) $$\begin{array}{c} P \longrightarrow Q \\ Q \\ \hline \therefore P \end{array}$$

(d) $$\begin{array}{c} P \longleftrightarrow Q \\ Q \\ \hline \therefore P \end{array}$$

(e) $$\begin{array}{c} \sim (P \longrightarrow Q) \\ P \\ \hline \therefore \sim Q \end{array}$$

(f) $$\begin{array}{c} P \longrightarrow Q \\ P \longrightarrow (\sim Q) \\ \hline \therefore \sim P \end{array}$$

(g) $$\begin{array}{c} P \\ \sim P \\ \hline \therefore Q \end{array}$$

(For this last case, see Footnote 35, page 35.)

8. For each of the following, decide if it is valid or a fallacy. For those marked "(Prove)," offer a truth table proof or a manipulation of the hypotheses to justify your answer.

(a) $$\begin{array}{c} (P \wedge Q) \longrightarrow R \\ P \\ Q \\ \hline \therefore R \end{array}$$

(b) $$\begin{array}{c} (P \wedge Q) \longrightarrow R \\ P \\ \sim R \\ \hline \therefore \sim Q \end{array}$$

(c) $$\begin{array}{c} (P \wedge Q) \longrightarrow R \\ Q \\ \sim P \\ \hline \therefore \sim R \text{ (Prove)} \end{array}$$

(d) $$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow R \\ \sim R \\ \hline \therefore \sim P \text{ (Prove)} \end{array}$$

(e) $$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow R \\ \sim P \\ \hline \therefore \sim R \end{array}$$

(f) $$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow R \\ R \longrightarrow P \\ \hline \therefore (P \longleftrightarrow R) \end{array}$$

(g) $$\begin{array}{c} P \vee Q \vee R \\ \sim P \\ \sim R \\ \hline \therefore Q \text{ (Prove)} \end{array}$$

(h) $$\begin{array}{c} P \longrightarrow Q \\ Q \longrightarrow (\sim R) \\ R \\ \hline \therefore \sim P \end{array}$$

(i) $$\begin{array}{c} P \vee Q \vee R \\ P \longrightarrow Q \\ \sim Q \\ \hline \therefore R \end{array}$$

(j) $$\begin{array}{c} P \longrightarrow (Q \wedge R) \\ \sim R \\ \hline \therefore \sim P \text{ (Prove)} \end{array}$$

(k) $$\begin{array}{c} P \longrightarrow (Q \vee R) \\ P \\ \sim R \\ \hline \therefore Q \end{array}$$

(l) $$\begin{array}{c} (\sim P) \longrightarrow (\sim Q) \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

(m) $$\begin{array}{c} (P \wedge Q) \rightarrow R \\ S \rightarrow P \\ U \rightarrow Q \\ S \wedge U \\ \hline \therefore R \end{array}$$

(n) $$\begin{array}{c} (P \wedge Q) \rightarrow R \\ S \rightarrow P \\ U \rightarrow Q \\ (\sim S) \wedge (\sim U) \\ \hline \therefore \sim R \end{array}$$

(o)
$$\begin{array}{c} (P \wedge Q) \to R \\ S \to P \\ U \to Q \\ \sim R \\ \hline \therefore\ (\sim S) \vee (\sim U) \end{array}$$

(p)
$$\begin{array}{c} \mathcal{F} \\ \hline \therefore P \end{array}$$

(q)
$$\begin{array}{c} P \\ \hline \therefore \mathcal{T} \end{array}$$

(r)
$$\begin{array}{c} \mathcal{T} \longrightarrow P \\ \hline \therefore P \end{array}$$

(s)
$$\begin{array}{c} P \longrightarrow \mathcal{F} \\ \hline \therefore \sim P \end{array}$$

(t)
$$\begin{array}{c} P \longrightarrow R \\ R \longrightarrow P \\ \hline \therefore P \longleftrightarrow R \text{ (Prove)} \end{array}$$

(u)
$$\begin{array}{c} P \longleftrightarrow R \\ Q \longleftrightarrow R \\ \hline \therefore P \longleftrightarrow R \end{array}$$

## 1.4 Quantifiers and Sets

In this section we introduce quantifiers, which form the last class of logic symbols we will consider in this text. To use quantifiers, we also need some notions and notation from set theory. This section introduces sets and quantifiers to the extent required for our study of calculus here. For the interested reader, Section 1.5 will extend this introduction, though even with that section we would be only just beginng to delve into these topics if studying them for their own sakes. Fortunately what we need of these topics for our study of calculus is contained in this section.

### 1.4.1 Sets

Put simply, a set is a collection of objects, which are then called *elements* or *members* of the set. We give sets names just as we do variables and statements. For an example of the notation, consider a set $A$ defined by

$$A = \{2, 3, 5, 7, 11, 13, 17\}.$$

We usually define a particular set by describing or listing the elements between "curly braces" $\{\ \}$ (so the reader understands it is indeed a *set* we are discussing). The defining of $A$ above was accomplished by a complete listing, but some sets are too large for that to be possible, let alone practical. As an alternative, the set $A$ above can also be written

$$A = \{x \mid x \text{ is a prime number less than } 18\}.$$

The above equation is usually read, "$A$ is the set of all $x$ such that $x$ is a prime number less than 18." Here $x$ is a "dummy variable," used only briefly to describe the set.[45] Sometimes it is convenient to simply write

$$A = \{\text{prime numbers between 2 and 17, inclusive}\}.$$

(Usually "inclusive" is meant by default, so here we would include 2 and 17 as possible elements, if they also fit the rest of the description.) Of course there are often several ways of describing a list of items. For instance, we can replace "between 2 and 17, inclusive" with "less than 18," as before.

Often an ellipsis "$\cdots$" is used when a pattern should be understood from a partial listing. This is particularly useful if a complete listing is either impractical or impossible. For instance, the set $B$ of integers from 1 to 100 could be written

$$B = \{1, 2, 3, \cdots, 100\}.$$

To note that an object is in a set, we use the symbol $\in$. For instance we may write $5 \in B$, read "5 is an element of $B$." To indicate concisely that 5, 6, 7 and 8 are in $B$, we can write $5, 6, 7, 8 \in B$.

Just as we have use for zero in addition, we also define the *empty set*, or *null set* as the set which has no elements. We denote that set $\varnothing$. Note that $x \in \varnothing$ is always false, i.e.,

$$x \in \varnothing \iff \mathcal{F},$$

because it is impossible to find any element of any kind inside $\varnothing$. We will revisit this set repeatedly in the optional, more advanced Section 1.5.

[45] "Dummy variables" are also used to describe the actions of functions, as in $f(x) = x^2 + 1$. In this context, the function is considered to be the *action* of taking an input number, squaring it, and adding 1. The $x$ is only there so we can easily trace the action on an arbitrary input. We will revisit functions later.

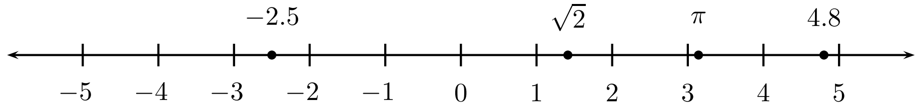



**Figure 1.2:** The number line representing the set $\mathbb{R}$ of real numbers, with a few points plotted. On this graph, the hash marks fall at the integers.

Of course for calculus we are mostly interested in sets of numbers. While not the most important, the following three sets will occur from time to time in this text:

$$\underline{\text{Natural Numbers}^{46}:}\qquad \mathbb{N} = \{1, 2, 3, 4, \cdots\}, \tag{1.67}$$

$$\underline{\text{Integers:}}\qquad \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}, \tag{1.68}$$

$$\underline{\text{Rational Numbers:}}\qquad \mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; (p, q \in \mathbb{Z}) \wedge (q \neq 0) \right\}. \tag{1.69}$$

Here we again use the ellipsis to show that the established pattern continues forever in each of the cases $\mathbb{N}$ and $\mathbb{Z}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are examples of *infinite* sets, i.e., sets that do not have a finite number of elements. The rational numbers are those which are *ratios* of integers, except that division by zero is not allowed, for reasons we will consider later.[47]

For calculus the most important set is the set $\mathbb{R}$ of *real numbers*, which cannot be defined by a simple listing or by a simple reference to $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$. One intuitive way to describe the real numbers is to consider the horizontal *number line*, where geometric points on the line are represented by their *displacements* (meaning distances, but counted as positive if to the right and negative if to the left) from a fixed point, called the *origin* in this context. That fixed point is represented by the number 0, since the fixed point is a displacement of zero units from itself. In Figure 1.2 the number line representation of $\mathbb{R}$ is shown. Hash marks at convenient intervals are often included. In this case, they are at the integers. The arrowheads indicate the number line is an actual line and thus infinite in both directions. The points $-2.5$ and $4.8$ on the graph are not integers, but are rational numbers, since they can be written $-25/10 = -5/2$, and $48/10 = 24/5$, respectively. The points $\sqrt{2}$ and $\pi$ are real, but not rational, and so are called *irrational.* To summarize,

**Definition 1.4.1** *The set of all* **real** *numbers is the set* $\mathbb{R}$ *of all possible displacements, to the right or left, of a fixed point* 0 *on a line. If the displacement is to the right, the number is the positive distance from* 0*. If to the left, the number is the negative of the distance from* 0*.*[48]

Thus

$$\mathbb{R} = \{\text{displacements from 0 on the number line}\}. \tag{1.70}$$

This is not a rigorous definition, not least because "right" and "left" require a fixed perspective. Even worse, the definition is really a kind of "circular reasoning," since we are effectively defining

[46] The natural numbers are also called *counting* numbers in some texts.

[47] For a hint, think about what should be $x = 1/0$. If we multiply both sides by zero, we might think we get $0x = (1/0) \cdot 0$, giving $0 = 1$, which is absurd. In fact there was no such $x$, so $x = 1/0 \rightarrow 0 = 1$, which is of the form $P \longrightarrow \mathcal{F}$ which we may recall to be equivalent to $\sim P$.

[48] It should be noted that we have to choose a direction to call "right," the other then being "left." It will depend upon our perspective. When we look at the Cartesian Plane, the horizontal axis measures displacements as right (positive horizontal) or left (negative horizontal), and the vertical axis measures displacements as upward (positive vertical) or downward (negative vertical). In that context the *origin* is where the axes intersect.

the number line in terms of $\mathbb{R}$, and then defining $\mathbb{R}$ in terms of (displacements on) the number line. We will give a more rigorous definition in Chapter 2 for the interested reader. For now this should do, since the number line is a simple and intuitive image.

### 1.4.2 Quantifiers

The three quantifiers used by nearly every professional mathematician are as follow:

| | | |
|---|---|---|
| **universal quantifier:** | $\forall$, | *read, "for all," or "for every;"* |
| **existential quantifier:** | $\exists$, | *read, "there exists;"* |
| **uniqueness quantifier:** | $!$, | *read, "unique."* |

The first two are of equal importance, and far more important than the third which is usually only found after the second. Quantified statements are usually found in forms such as:

$$\begin{aligned}
&(\forall x \in S)P(x), && \text{i.e., for all } x \in S,\ P(x) \text{ is true;}\\
&(\exists x \in S)P(x), && \text{i.e., there exists an } x \in S \text{ such that } P(x) \text{ is true;}\\
&(\exists! x \in S)P(x), && \text{i.e., there exists a unique (exactly one) } x \in S \text{ such that}\\
& && \quad P(x) \text{ is true.}
\end{aligned}$$

Here $S$ is a set and $P(x)$ is some statement about $x$. The meanings of these quickly become straightforward. For instance, consider

$$\begin{aligned}
(\forall x \in \mathbb{R})(x + x = 2x):&\ \text{for all } x \in \mathbb{R},\ x + x = 2x;\\
(\exists x \in \mathbb{R})(x + 2 = 2):&\ \text{there exists (an) } x \in \mathbb{R} \text{ such that } x + 2 = 2;\\
(\exists! x \in \mathbb{R})(x + 2 = 2):&\ \text{there exists a unique } x \in \mathbb{R} \text{ such that } x + 2 = 2.
\end{aligned}$$

All three quantified statements above are true. In fact they are true under any circumstances, and can thus be considered tautologies. Unlike unquantified statements $P, Q, R$, etc., from our first three sections, a quantified statement is either true always or false always, and is thus, for our purposes, equivalent to either $\mathcal{T}$ or $\mathcal{F}$. Each has to be analyzed on its face, based upon known mathematical principles; we do not have a brute-force mechanism analogous to truth tables to analyze these systematically.[49] For a couple more short examples, consider the following cases from algebra which should be clear enough:

$$\begin{aligned}
(\forall x \in \mathbb{R})(0 \cdot x = 0) &\iff \mathcal{T};\\
(\exists x \in \mathbb{R})(x^2 = -1) &\iff \mathcal{F}.
\end{aligned}$$

The optional advanced section shows how we can still find equivalent or implied statements from quantified statements in many circumstances.

### 1.4.3 Statements with Multiple Quantifiers

Many of the interesting statements in mathematics contain more than one quantifier. To illustrate the mechanics of multiply quantified statements, we will first turn to a more worldly setting. Consider the following sets:

$$\begin{aligned}
M &= \{\text{men}\},\\
W &= \{\text{women}\}.
\end{aligned}$$

[49] This is part of what makes quantified statements interesting!

In other words, $M$ is the set of all men, and $W$ the set of all women. Consider the statement[50]

$$(\forall m \in M)(\exists w \in W)[w \text{ loves } m]. \tag{1.71}$$

Set to English, (1.71) could be written, "for every man there exists a woman who loves him."[51] So if (1.71) is true, we can in principle arbitrarily choose a man $m$, and then know that there is a woman $w$ who loves him. It is important that the man $m$ was quantified *first*. A common syntax that would be used by a logician or mathematician would be to say here that, once our choice of a man is *fixed*, we can in principle find a woman who loves him. Note that (1.71) allows that different men may need different women to love them, and also that a given man may be loved by more than (but not less than) one woman.

Alternatively, consider the statement

$$(\exists w \in W)(\forall m \in M)[w \text{ loves } m]. \tag{1.72}$$

A reasonable English interpretation would be, "there exists a woman who loves every man." Granted that is a summary, for the word-for-word English would read more like, "there exists a woman such that, for every man, she loves him." This says something very different from (1.71), because that earlier statement does not assert that we can find a woman who, herself, loves every man, but that for each man there is a woman who loves him.[52]

We can also consider the statement

$$(\forall m \in M)(\forall w \in W)[w \text{ loves } m]. \tag{1.73}$$

This can be read, "for every man and every woman, the woman loves the man." In other words, every man is loved by every woman. In this case we can reverse the order of quantification:

$$(\forall w \in W)(\forall m \in M)[w \text{ loves } m]. \tag{1.74}$$

In fact, if the two quantifiers are the same type—both universal or both existential—then the order does not matter. Thus

$$\begin{aligned}(\forall m \in M)(\forall w \in W)[w \text{ loves } m] &\iff (\forall w \in W)(\forall m \in M)[w \text{ loves } m],\\ (\exists m \in M)(\exists w \in W)[w \text{ loves } m] &\iff (\exists w \in W)(\exists m \in M)[w \text{ loves } m].\end{aligned}$$

In both representations in the existential statements, we are stating that there is at least one man and one woman such that she loves him. In fact that above equivalence is also valid if we replace $\exists$ with $\exists!$, though it would mean then that there is exactly one man and exactly one woman such that the woman loves the man, but we will not delve too deeply into uniqueness here.

Note that in cases where the sets are the same, we can combine two similar quantifications into one, as in

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = y + x] \iff (\forall x, y \in \mathbb{R})[x + y = y + x]. \tag{1.75}$$

Similarly with existence.

[50] Note that this is of the form $(\forall m \in M)(\exists w \in W)P(m, w)$, that is, the statement $P$ says something about both $m$ and $w$. We will avoid a protracted discussion of the difference between statements regarding one variable object—as in $P(x)$ from our previous discussion—and statements which involve more than one as in $P(m, w)$ here. Statements of multiple (variable) quantities will recur in subsequent examples.

[51] At times it seems appropriate to translate "$\forall$" as "for all," and at other times it seems better to translate it as "for every." Both mean the same.

[52] We do not pretend to know the truth values of either (1.71) or (1.72).

However, we repeat the point at the beginning of the subsection, which is that the order does matter if the types of quantification are different.

For another, short example which is algebraic in nature, consider

$$(\forall x \in \mathbb{R})(\exists K \in \mathbb{R})(x = 2K). \qquad \text{(True.)} \tag{1.76}$$

This is read, "for every $x \in \mathbb{R}$, there exists $K \in \mathbb{R}$ such that $x = 2K$." That $K = x/2$ exists (and is actually unique) makes this true, while it would be false if we were to reverse the order of quantification:

$$(\exists K \in \mathbb{R})(\forall x \in \mathbb{R})(x = 2K). \qquad \text{(False.)} \tag{1.77}$$

Statement (1.77) claims (erroneously) that there exists $K \in \mathbb{R}$ so that, for every $x \in \mathbb{R}$, $x = 2K$. That is impossible, because no value of $K$ is half of *every* real number $x$. For example the value of $K$ which works for $x = 4$ is not the same as the value of $K$ which works for $x = 100$.

### 1.4.4 Detour: Uniqueness as an Independent Concept

We will have occasional statements in the text which include uniqueness. However, most of those will not require us to rewrite the statements in ways which require actual manipulation of the uniqueness quantifier. Still, it is worth noting a couple of interesting points about this quantifier.

First we note that uniqueness can be formulated as a separate concept from existence, interestingly instead requiring the universal quantifier.

**Definition 1.4.2 Uniqueness** *is the notion that if $x_1, x_2 \in S$ satisfy the same particular statement $P(\ )$, then they must in fact be the same object. That is, if $x_1, x_2 \in S$ and $P(x_1)$ and $P(x_2)$ are true, then $x_1 = x_2$. This may or may not be true, depending upon the set $S$ and the statement $P(\ )$.*

Note that there is the vacuous case where nothing satisfies the statement $P(\ )$, in which case the uniqueness of any such hypothetical object is proved but there is actually no existence. Consider the following, symbolic representation of the uniqueness of an object $x$ which satisfies $P(x)$:[53]

$$(\forall x, y \in S)[(P(x) \wedge P(y)) \longrightarrow x = y]. \tag{1.78}$$

Finally we note that a proof of a statement such as $(\exists! x \in S)P(x)$ is thus usually divided into two separate proofs:

(1) *Existence:* $(\exists x \in S)P(x)$;

(2) *Uniqueness:* $(\forall x, y \in S)[(P(x) \wedge P(y)) \longrightarrow x = y]$.

For example, in the next chapter we rigorously, axiomatically define the set of real numbers $\mathbb{R}$. One of the axioms[54] defining the real numbers is the existence of an additive identity:

$$(\exists z \in \mathbb{R})(\forall x \in \mathbb{R})(z + x = x). \tag{1.79}$$

[53]The above statement indeed says that any two elements $x, y \in S$ which both satisfy $P$ must be the same. Note that we use a single arrow here, because the statement between the brackets [ ] is not likely to be a tautology, but may be true for enough cases for the entire quantified statement to be true. Indeed, the symbols $\Longrightarrow$ and $\Longleftrightarrow$ belong *between* quantified statements, not *inside* them.

[54]Recall that an *axiom* is an assumption, usually self-evident, from which we can logically argue towards theorems. Axioms are also known as *postulates*. If we attempt to argue only using "pure logic" (as a mathematician does when developing theorems, for instance), it eventually becomes clear that we still need to make some assumptions because one can not argue "from nothing." Indeed, some "starting points" from which to argue towards the conclusions are required. These are then called axioms.

The word "axiomatic" is often used colloquially to mean clearly evident and therefore not requiring proof. In

In fact it follows quickly that such a "$z$" must be unique, so we have

$$(\exists! z \in \mathbb{R})(\forall x \in \mathbb{R})(z + x = x). \tag{1.80}$$

To prove (1.80), we need to prove (1) existence, and (2) uniqueness. In this setting, the existence is an axiom so there is nothing to prove. We turn then to the uniqueness. A proof is best written in prose, but it is based upon proving that the following is true:

$$(\forall z_1, z_2 \in \mathbb{R})[(z_1 \text{ an additive identity}) \wedge (z_2 \text{ an additive identity}) \longrightarrow z_1 = z_2].$$

Now we prove this. Suppose $z_1$ and $z_2$ are additive identities, i.e., they can stand in for $z$ in (1.79), which could also read $(\exists z \in \mathbb{R})(\forall x \in \mathbb{R})(x = z + x)$. Note the order there, where the identity $z$ (think "zero") is placed on the left of $x$ in the equation $x = z + x$. So, assuming $z_1, z_2$ are additive identities, we have:

$$\begin{aligned} z_1 &= z_2 + z_1 && \text{(since } z_2 \text{ is an additive identity)} \\ &= z_1 + z_2 && \text{(since addition is commutative—order is irrelevant)} \\ &= z_2 && \text{(since } z_1 \text{ is an additive identity).} \end{aligned}$$

This argument showed that if $z_1$ and $z_2$ are any real numbers which act as additive identities, then $z_1 = z_2$. In other words, if there are any additive identities, there must be only one. Of course, assuming its existence we call that unique real number *zero*. (It should be noted that the commutativity used above is another axiom of the real numbers. We will list fourteen in all.)

The distinction between existence and uniqueness of an object with some property $P$ is often summarized as follows:

(1) Existence asserts that there is at least one such object.

(2) Uniqueness asserts that there is at most one such object.

If both hold, then there is exactly one such object.

### 1.4.5 Negating Universally and Existentially Quantified Statements

For statements with a single universal or existential quantifier, we have the following negations.

$$\sim [(\forall x \in S)P(x)] \iff (\exists x \in S)[\sim P(x)], \tag{1.81}$$

$$\sim [(\exists x \in S)P(x)] \iff (\forall x \in S)[\sim P(x)], \tag{1.82}$$

The left side of (1.81) states that it is not the case that $P(x)$ is true for all $x \in S$; the right side states that there is an $x \in S$ for which $P(x)$ is false. We could ask when is it a lie that for all $x$, $P(x)$ is true? The answer is when there is an $x$ for which $P(x)$ is false, i.e., $\sim P(x)$ is true.

The left side of (1.82) states that it is not the case that there exists an $x \in S$ so that $P(x)$ is true; the right side says that $P(x)$ is false for all $x \in S$. When is it a lie that there is an $x$ making $P(x)$ true? When $P(x)$ is false for all $x$.

Thus when we negate such a statement as $(\forall x)P(x)$ or $(\exists x)P(x)$, we change $\forall$ to $\exists$ or vice-versa, and negate the statement after the quantifiers.

---

fact that is not always the case with mathematical axioms. Indeed, the postulates required for defining the real numbers seem rather strange at first. They were, in fact, developed to be a minimal number of assumptions required to give the real numbers their apparent properties which could be observed. In that case, it seemed we worked towards a foundation, after seeing the outer structure. A similar phenomenon can be seem in Einstein's Special Relativity, where his two simple—yet at the time quite counterintuitive—axioms were able to completely replace a much larger set of postulates required to explain many of the electromagnetic phenomena discovered early during his time, and predict many new phenomena that were later observed.

**Example 1.4.1** *Negate* $(\forall x \in S)[P(x) \longrightarrow Q(x)]$.
<u>*Solution*</u>*: We will need* (1.21), page 22, *namely* $\sim (P \longrightarrow Q) \iff P \wedge (\sim Q)$.

$$\begin{aligned} \sim [(\forall x \in S)(P(x) \longrightarrow Q(x))] &\iff (\exists x \in S)[\sim (P(x) \longrightarrow Q(x))] \\ &\iff (\exists x \in S)[P(x) \wedge (\sim (Q(x)))]. \end{aligned}$$

The above example should also be intuitive. To say that it is not the case that, for all $x \in S$, $P(x) \longrightarrow Q(x)$ is to say there exists an $x$ so that we do have $P(x)$, but not the consequent $Q(x)$.

**Example 1.4.2** *Negate* $(\exists x \in S)[P(x) \wedge Q(x)]$.
<u>*Solution*</u>*: Here we use* $\sim (P \wedge Q) \iff (\sim P) \vee (\sim Q)$, *so we can write*

$$\sim [(\exists x \in S)(P(x) \wedge Q(x))] \iff (\forall x)[(\sim P(x)) \vee (\sim (Q(x)))].$$

This last example shows that if it is not the case that there exists an $x \in S$ so that $P(x)$ and $Q(x)$ are both true, that is the same as saying that for all $x$, either $P(x)$ is false or $Q(x)$ is false.

### 1.4.6 Negating Statements Containing Mixed Quantifiers

Here we simply apply (1.81) and (1.82) two or more times, as appropriate. For a typical case of a statement first quantified by $\forall$, and then be $\exists$, we note that we can group these as follows:[55]

$$(\forall x \in R)(\exists y \in S)P(x,y) \iff (\forall x \in R)[(\exists y \in S)P(x,y)].$$

(Here "$R$" is another set, not to be confused with the set of real numbers $\mathbb{R}$.) Thus

$$\begin{aligned} \sim [(\forall x \in R)(\exists y \in S)P(x,y)] &\iff \sim \{(\forall x \in R)[(\exists y \in S)P(x,y)]\} \\ &\iff (\exists x \in R)\{\sim [(\exists y \in S)P(x,y)]\} \\ &\iff (\exists x \in R)(\forall y \in S)[\sim P(x,y)]. \end{aligned}$$

Ultimately we have, in turn, the $\forall$'s become $\exists$'s, the $\exists$'s become $\forall$'s, the variables are quantified in the same order as before, and finally the statement $P$ is replaced by its negation $\sim P$. The pattern would continue no matter how many universal and existential quantifiers arise. (The uniqueness quantifier is left for the exercises.) To summarize for the case of two quantifiers,

$$\sim [(\forall x \in R)(\exists y \in S)P(x,y)] \iff (\exists x \in R)(\forall y \in S)[\sim P(x,y)] \tag{1.83}$$
$$\sim [(\exists x \in R)(\forall y \in S)P(x,y)] \iff (\forall x \in R)(\exists y \in S)[\sim P(x,y)]. \tag{1.84}$$

**Example 1.4.3** *Consider the following statement, which is false:*

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[xy = 1].$$

*One could say that the statement says every real number $x$ has a real number reciprocal $y$. This is false, but before that is explained, we compute the negation which must be true:*

$$\sim [(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)] \iff (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1).$$

*Indeed, there exists such an $x$, namely $x = 0$, such that $xy \neq 1$ for all $y$.*[56]

[55] This may not be the most transparent fact, and indeed there are somewhat deep subtleties involved, but eventually this should be clear. The subtleties lie in the idea that once a variable is quantified, it is *fixed* for that part of the statement which follows it. For instance, that part $(\exists y \in S)P(x,y)$ treats $x$ as if it were "constant."

[56] Note that in using English, the quantification often follows after the variable quantified, as in Example 1.4.3 above. That can become quite confusing when statements get complicated. Indeed, much of the motivation of this section is so that we can use the notation to, in essence, diagram the logic of such statements, and analyze them to see if they may be false (by seeing if their negations ring true).

In the above, we borrowed one of the many convenient mathematical notations for the negations of various symbols. Some common negations follow:

$$\begin{aligned}
\sim (x = y) &\iff x \neq y,\\
\sim (x < y) &\iff x \geq y,\\
\sim (x \leq y) &\iff x > y,\\
\sim (x \in S) &\iff x \notin S.
\end{aligned}$$

Of course we can negate both sides of any one of these and get, for example, $x \in S \iff \sim (x \notin S)$. Reading one of these backwards, we can have $\sim (x \geq y) \iff x < y$.

## Exercises

1. Consider the sets

$$\begin{aligned}
P &= \{\text{prisons}\},\\
M &= \{\text{methods of escape}\}.
\end{aligned}$$

   For each of the following, write a short English version of the given statement.

   (a) $(\forall p \in P)(\exists m \in M)[m \text{ will get you out of } P]$
   (b) $(\exists m \in M)(\forall p \in P)[m \text{ will get you out of } P]$
   (c) $(\exists p \in P)(\forall m \in M)[m \text{ will get you out of } P]$
   (d) $(\forall m \in M)(\exists p \in P)[m \text{ will get you out of } P]$
   (e) $(\exists m \in M)(\exists p \in P)[m \text{ will get you out of } P]$
   (f) $(\forall m \in M)(\forall p \in P)[m \text{ will get you out of } P]$

2. For parts (a)–(d) above, write the negation of the statment both symbolically and in English.

3. Write negations for each of the following. If in the process a logical statement within the quantified statement is negated, write the negation in the clearest possible form. For instance, instead of writing $\sim (P \rightarrow Q)$, write $P \wedge (\sim Q)$. Similarly instead of writing $\sim (x > y)$ write $x \leq y$.

   (a) $(\forall x \in R)[x \in S]$
   (b) $(\forall x \in R)(\exists y \in S)[y \leq x]$
   (c) $(\forall x, y \in R)(\exists r, t \in S)[rx + ty = 1]$
   Hint: This can also be written $(\forall x \in R)(\forall y \in R)(\exists r \in S)(\exists t \in S)[rx + ty = 1]$.
   (d) $(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R}) \left[|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right]$
   Hint: consider $P : |x - a| < \delta$ and $Q : |f(x) - f(a)| < \varepsilon$. Then consider the usual negation of $P \rightarrow Q$, with these statements inserted literally, and then rewrite it in a more "understandable" way.

4. For each of the following, write the negation of the statement and decide which is true, the original statement or its negation.
   (a) $(\exists x \in \mathbb{R})(x^2 < 0)$
   (b) $(\forall x \in \mathbb{R})(|-x| \neq x)$
   (c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y = 2x + 1)$
5. Consider $(\forall c \in C)(\exists b \in B)(b \text{ would buy } c)$. Here $C = \{\text{cars}\}$ and $B = \{\text{buyers}\}$.
   (a) Write in English what this statement says.
   (b) Write in English what the negation of the statement should be.
   (c) Write in symbolic logic what the negation of the statement should be.
6. Consider the statement $(\forall x, y \in \mathbb{R})(x < y \longrightarrow x^2 < y^2)$.
   (a) Write the negation of this statement.
   (b) In fact it is the negation that is true. Can you explain why?
7. Using the fact that

$$(\exists! x \in S)P(x) \iff \underbrace{\Big[(\exists x \in S)P(x)\Big]}_{\text{existence}} \bigwedge \underbrace{\Big[(\forall x, y \in S)[(P(x) \wedge P(y)) \longrightarrow x = y]\Big]}_{\text{uniqueness}},$$

compute the form of the negation of the unique existence:

$$\sim [(\exists! x \in S)P(x)].$$

## 1.5 Sets Proper

In this section we introduce set theory in its own right. We also apply the earlier symbolic logic to the theory of sets (rather than vice-versa). We also approach set theory visually and intuitively, while simultaneously introducing all the set-theoretic notation we will use throughout the text. To begin we make the following definition:

**Definition 1.5.1** *A* **set** *is a well-defined collection of objects.*

By well-defined, we mean that once we define the set, the objects contained in the set are totally determined, and so any given object is either in the set or not in the set. We might also note that in a sense a set is defined (or *determined*) by its elements; sets which are different collections of elements are different sets, while sets with exactly the same elements are the same set. We can also define equality by means of quantifiers:

**Definition 1.5.2** *Given two sets $A$ and $B$, we defined the statement $A = B$ as being equivalent to the statement $(\forall x)[(x \in A) \longleftrightarrow (x \in B)]$:*

$$A = B \iff (\forall x)[(x \in A) \longleftrightarrow (x \in B)]. \tag{1.85}$$

*If we allow ourselves to understand that $x$ is quantified universally (that is, we assume "$(\forall x)$" is understood) unless otherwise stated, we can write, instead of $A = B$, that $x \in A \iff x \in B$.*

When we say a set is well-defined we also mean that once defined the set is *fixed*, and does not change. If elements can be listed in a table (finite or otherwise),[57] then the order we list the elements is not relevant; sets are defined by exactly which objects are elements, and which are not. Moreover, it is also irrelevant if objects are listed more than once in the set, such as when we list $\mathbb{Q} = \{x \mid x = p/q,\ p, q \in \mathbb{Z},\ q \neq 0\}$. In that definition $2 = 2/1 = 4/2 = 6/3$ is "listed" infinitely many times, but it is simply one element of the set of rational numbers $\mathbb{Q}$. While it actually is possible to "list" the elements of $\mathbb{Q}$ if we allow for the elipsis $(\cdots)$, it is more practical to describe the set, as we did, using some *defining property* of its elements (here they were ratios of integers, without dividing by zero), as long as it is exactly those elements in the set—no more and no fewer—which share that property. One usually uses a "dummy variable" such as $x$ and then describes what properties all such $x$ in the set should have. We could have just as easily used $z$ or any other variable.[58]

### 1.5.1 Subsets and Set Equality

When all the elements of a set $A$ are also elements of another set $B$, we say $A$ is a *subset* of $B$. To express this in set notation, we write $A \subseteq B$. In this case we can also take another perspective, and say $B$ is a *superset* of $A$, written $B \supseteq A$. Both symbols represent types of *set inclusions*, i.e., they show one set is contained in another.

A useful graphical device which can illustrate the notion that $A \subseteq B$ and other set relations is the *Venn Diagram*, as in Figure 1.3. There we see a visual representation of what it means for $A \subseteq B$. The sets are represented by enclosed areas in which we imagine the elements reside. In each representation given in Figure 1.3, all the elements inside $A$ are also inside $B$.

[57] Note that not all sets can be listed in a table, even if it is infinitely long. We can list $\mathbb{N} = \{1, 2, 3, \cdot\}$ and $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$, and even with a little ingenuity list the elements of $\mathbb{Q}$, but we cannot do so with $\mathbb{R}$ or $\mathbb{R} - \mathbb{Q}$. Those sets which can be listed in a table are called *countable*, and the others *uncountable*. All sets with a finite number of elements are also countable. Of the others, some are countably infinite, and the others are uncountably infinite (or simply uncountable, as the "infinite" in "uncountably infinite" is redundant).

[58] Of course we would not use fixed elements of the set as "variables," which they are not since each has a unique identity.

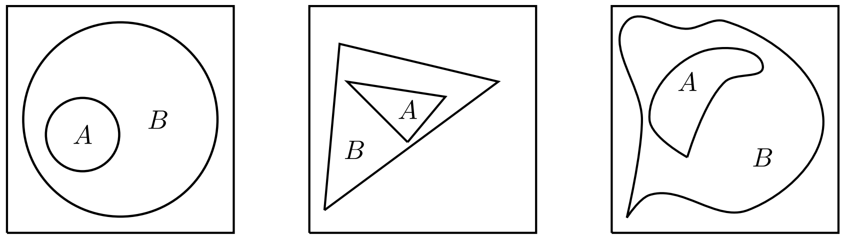



**Figure 1.3:** Three possible Venn Diagrams illustrating $A \subseteq B$. (Note that in the first figure, for example, $B$ is the set of all elements within the interior of the larger circle.) What is important is that all elements of $A$ are necessarily contained in $B$ as well. We do not necessarily know "where" in $A$ are the elements of $A$, except that they are in the area which is marked by $A$. Since the area in $A$ is also in $B$, we know the elements of $A$ must also be contained in $B$ in the illustrations above.

Using symbolic logic, we can *define* subsets, and the notation, as follows:

$$A \subseteq B \iff (\forall x)(x \in A \longrightarrow x \in B). \tag{1.86}$$

The role of the implication which is the main feature of (1.86) should seem intuitive. Perhaps less intuitive are some of the statements which are therefore logically equivalent to (1.86):

$$\begin{aligned} A \subseteq B &\iff (\forall x)(x \in A \longrightarrow x \in B) \\ &\iff (\forall x)\left[(\sim (x \in A)) \vee (x \in B)\right] \\ &\iff (\forall x)\left[(x \notin A) \vee (x \in B)\right], \end{aligned}$$

which uses the fact that $P \longrightarrow Q \iff (\sim P) \vee Q$, and

$$\begin{aligned} A \subseteq B &\iff (\forall x)\left[(\sim (x \in B)) \longrightarrow (\sim (x \in A))\right] \\ &\iff (\forall x)\left[(x \notin B) \longrightarrow (x \notin A)\right] \end{aligned}$$

which uses the contrapositive $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$. Note that we used the shorthand notation $\sim (x \in A) \iff x \notin A$. With the definition (1.86) we can quickly see two more, technically interesting facts about subsets:

**Theorem 1.5.1** *For any sets $A$ and $B$, the following hold true:*

$$A \subseteq A, \qquad \textit{and} \tag{1.87}$$
$$A = B \iff (A \subseteq B) \wedge (B \subseteq A). \tag{1.88}$$

Now we take a moment to remind ourselves of what is meant by *theorem:*

**Definition 1.5.3** *A* **theorem** *is a statement which we know to be true because we have a proof of it. We can therefore accept it as a tautology.*

A theorem's scope may be very limited (the above theorem only applies to sets and subsets as we have defined them.) Furthermore, a theorem's scope and "truth" depends upon the axiomatic system upon which it rests, such the definitions we gave our symbolic logic symbols (which might not have always been completely obvious to the novice, as in our definitions of "$\vee$" and "*longrightarrow*"). For another example there is Euclidean geometry, the theroems of which

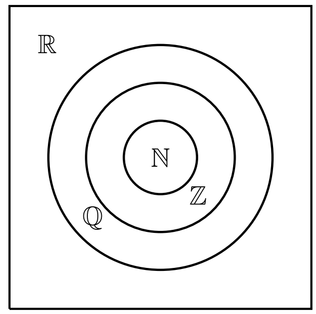



**Figure 1.4:** Venn Diagram illustrating $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

rest upon Euclid's Postulates (or axioms, or original assumptions), while other geometric systems begin with different postulates.

Nonetheless once we have the definitions and postulates one can say that a theorem is a statement which is always true (demonstrated by some form of proof), and in fact therefore equivalent to $\mathcal{T}$ (introduced on page 26). We will use that fact in the proof of (1.87), but for (1.88) we will instead demonstrate the validity of the equivalence ( $\iff$ ). For the first statement's proof, we have

$$A \subseteq A \iff (\forall x)[(x \in A) \longrightarrow (x \in A)] \iff \mathcal{T}.$$

Note that the above proof is based upon the fact that $P \longrightarrow P$ is a tautology (i.e., equivalent to $\mathcal{T}$). A glance at a Venn Diagram with a set $A$ can also convince one of this fact, that any set is a subset of itself. For the proof of (1.88) we offer the following:

$$\begin{aligned} A = B &\iff (\forall x)[(x \in A) \longleftrightarrow (x \in B)] \\ &\iff (\forall x)[((x \in A) \longrightarrow (x \in B)) \wedge ((x \in B) \longrightarrow (x \in A))] \\ &\iff \big[(\forall x)[(x \in A) \longrightarrow (x \in B)\big] \wedge \big[(\forall x)[(x \in B) \longrightarrow (x \in A)\big] \\ &\iff (A \subseteq B) \wedge (B \subseteq A), \text{ q.e.d.}^{59} \end{aligned}$$

A consideration of Venn diagrams also leads one to believe that for all the area in $A$ to be contained in $B$ and vice versa, it must be the case that $A = B$. That $A = B$ implies they are mutual subsets is perhaps easier to see.

Note that the above arguments can also be made with supersets instead of subsets, with $\supseteq$ replacing $\subseteq$ and $\longleftarrow$ replacing $\longrightarrow$.

One needs to be careful with quantifiers and symbolic logic, as is discussed later in Section **??**, but in what we did above the $(\forall x)$ effectively went along for the ride.

Of course, Venn Diagrams can accommodate more than two sets. For example, we can illustrate the chain of set inclusions

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \tag{1.89}$$

using a Venn Diagram, as in Figure 1.4. Note that this is a compact way of writing six different set inclusions: $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{N} \subseteq \mathbb{Q}$, $\mathbb{N} \subseteq \mathbb{R}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Z} \subseteq \mathbb{R}$, and $\mathbb{Q} \subseteq \mathbb{R}$.

[59] Latin, *quod erat demonstrandum*), the traditional ending of a proof meaning *that which was to be proved.*

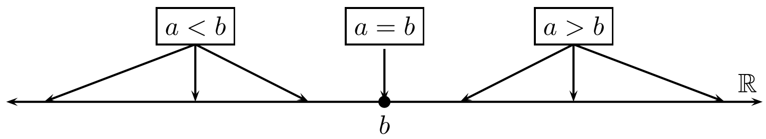



**Figure 1.5:** For any two real numbers $a$ and $b$, we have the three cases concerning their relative positions on the real line: $a < b$, $a = b$, $a > b$. Arrows indicate the possible positions of $a$ for the three cases.

### 1.5.2 Intervals and Inequalities in $\mathbb{R}$

The number line, which we will henceforth dub the *real line*, has an inherent order in which the numbers are arranged. Suppose we have two numbers $a, b \in \mathbb{R}$. Then the order relation between $a$ and $b$ has three possibilities, each with its own notation:

1. $a$ is to the left of $b$, written $a < b$ and spoken "$a$ is *less than* $b$."

2. $a$ is to the right of $b$, written $a > b$ and spoken "$a$ is *greater than* $b$."

3. $a$ is at the same location as $b$, written $a = b$ and spoken "$a$ equals $b$."

Figure 1.5 shows these three possibilities. Note that "less than" and "greater than" refer to relative positions on the real line, not how "large" or "small" the numbers are. For instance, $4 < 5$ but $-5 < -4$, though it is natural to consider $-5$ to be a "larger" number than $-4$. Similarly $-1000 < 1$.[60] Of course if $a < b \iff b > a$. We have further notation which describes when $a$ is left of or at $b$, and when $a$ is right of or at $b$:

4. $a$ is at or left of $b$, written $a \leq b$ and spoken "$a$ is less than or equal to $b$."

5. $a$ is at or right of $b$, written $a \geq b$ and spoken "$a$ is greater than or equal to $b$."

Using inequalities, we can describe *intervals* in $\mathbb{R}$, which are exactly the *connected* subsets of $\mathbb{R}$, meaning those sets which can be represented by darkening the real line at only those points which are in the subset, and where doing so can be theoretically accomplished without lifting our pencils as we darken. In other words, these are "unbroken" subsets of $\mathbb{R}$. Later we will see that intervals are subsets of particular interest in calculus.

Intervals can be classified as finite or infinite (referring to their lengths), and open, closed or half-open (referring to their "endpoints"). The finite intervals are of three types: closed, open and half-open. Intervals of these types, with real *endpoints* $a$ and $b$, where $a < b$ (though the idea extends to work with $a \leq b$) are shown below respectively by graphical illustration, in *interval notation*, and using earlier set-theoretic notation:

[60] Later we will refer to a number's *absolute size*, in which context we will describe $-1000$ as "larger" than 1.

| | | | |
|---|---|---|---|
| open: |  | $(a, b)$ | $\{x \in \mathbb{R} \mid a < x < b\}$ |
| closed: |  | $[a, b]$ | $\{x \in \mathbb{R} \mid a \le x \le b\}$ |
| half-open: |  | $[a, b)$ | $\{x \in \mathbb{R} \mid a \le x < b\}$ |
| half-open: |  | $(a, b]$ | $\{x \in \mathbb{R} \mid a < x \le b\}$ |

Note that $a < x < b$ is short for $(a < x) \wedge (x < b)$, i.e., $(x > a) \wedge (x < b)$. The others are similar.

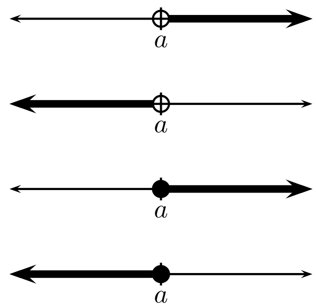
We will concentrate on the open and closed intervals in calculus. For the finite open interval above, we see that we do not include the endpoints $a$ and $b$ in the set, denoting this fact with parentheses in the interval notation and an "open" circle at each endpoint on the graph. What is crucial to calculus is that immediately surrounding any point $x \in (a, b)$ are only other points still inside the interval; if we pick a point $x$ *anywhere* in the interval $(a, b)$, we see that just left and just right of $x$ are only points in the interval. Indeed, we have to travel some distance—albeit possibly short—to leave the interval from a point $x \in (a, b)$. Thus no point inside of $(a, b)$ is on the boundary, and so each point in $(a, b)$ is "safely" on the interior of the interval. This will be crucial to the concepts of continuity, limits and (especially) derivatives later in the text.

For a closed interval $[a, b]$, we *do* include the endpoints $a$ and $b$, which are not surrounded by other points in the interval. For instance, immediately left of $a$ is outside the interval $[a, b]$, though immediately right of $a$ is on the interior.[61] We denote this fact with brackets in the interval notation, and a "closed" circle at each endpoint when we sketch the graph. Half-open (or half-closed) intervals are simple extensions of these ideas, as illustrated above.

For infinite intervals, we have either one or no endpoints. If there is an endpoint it is either not included in the interval or it is, the former giving an open interval and the latter a closed interval. An open interval which is infinite in one direction will be written $(a, \infty)$ or $(-\infty, a)$, depending upon the direction in which it is infinite. Here $\infty$ (infinity) means that we can move along the interval to the right "forever," and $-\infty$ means we can move left without end. For infinite closed intervals the notation is similar: $[a, \infty)$ and $(-\infty, a]$. The whole real line is also considered an interval, which we denote $\mathbb{R} = (-\infty, \infty)$.[62] When an interval continues *without bound* in a direction, we also darken the arrow in that direction. Thus we have the following:

[61] For a closed interval $[a, b]$, later we will sometimes refer to the *interior* of the interval, meaning all points whose immediate neighbors left and right are also in the interval. This means that the interior of $[a, b]$ is simply $(a, b)$.

[62] For technical reasons which will be partially explained later, $\mathbb{R}$ is considered to be both an open and a closed interval. Roughly, it is open because every point is interior, but closed because every point that can be approached as close as we want from the interior is contained in the interval. Those are the topological factors which characterize open and closed intervals as such. Topology as a subject is rarely taught before the junior level of college, or even graduate school, though advanced calculus usually includes some topology of $\mathbb{R}$.




| | | |
|---|---|---|
| open: | $(a, \infty)$ | $\{x \in \mathbb{R} \mid x > a\}$ |
| open: | $(-\infty, a)$ | $\{x \in \mathbb{R} \mid x < a\}$ |
| closed: | $[a, \infty)$ | $\{x \in \mathbb{R} \mid x \geq a\}$ |
| closed: | $(-\infty, a]$ | $\{x \in \mathbb{R} \mid x \leq a\}$ |

Note that we never use brackets to enclose an infinite "endpoint," since $-\infty, \infty$ are not actual boundaries but rather are concepts of unending continuance. Indeed, $-\infty, \infty \notin \mathbb{R}$, i.e., they are not points on the real line, so they can not be boundaries of subsets of $\mathbb{R}$; there are no elements "beyond" them.

### 1.5.3 Most General Venn Diagrams

Before we get to the title of this subsection, we will introduce a notion which we will have occasional use for, which is the concept of *proper subset.*

**Definition 1.5.4** *If $(A \subseteq B) \wedge (A \neq B)$, we call $A$ a* **proper** *subset of $B$, and write $A \subset B$*[63].

Thus $A \subset B$ means $A$ is contained in $B$, but $A$ is not all of $B$. Note that $A \subset B \implies A \subseteq B$ (just as $P \wedge Q \implies P$). When we have that $A$ is a subset of $B$ and are not interested in emphasizing whether or not $A \neq B$ (or are not sure if this is true), we will use the "inclusive" notation $\subseteq$. In fact, the inclusive case is less complicated logically (just as $P \vee Q$ is easier than $P$ XOR $Q$) and so we will usually opt for it even when we do know that $A \neq B$. We mention the exclusive case here mainly because it is useful in explaining the most general Venn Diagram for two sets $A$ and $B$.

Of course it is possible to have two sets, $A$ and $B$, where neither is a subset of the other. Then $A$ and $B$ may share some elements, or no elements. In fact, for any given sets $A$ and $B$, exactly one of the following will be true:

**case 1:** $A = B$;

**case 2:** $A \subset B$, i.e., $A$ is a proper subset of $B$;

**case 3:** $B \subset A$, i.e., $B$ is a proper subset of $A$;

**case 4:** $A$ and $B$ share common elements, but neither is a subset of the other;

**case 5:** $A$ and $B$ have no common elements. In such a case the two sets are said to be *disjoint.*

Even if we do not know which of the five cases is correct, we can use a single illustration which covers all of these. That illustration is given in Figure 1.6, with the various regions labeled. (We will explain the meaning of $U$ in the next subsection.) To see that this covers all cases, we take them in turn:

[63] The notation has changed over the years. Many current texts use "$\subset$" the way we use "$\subseteq$" here. This is unfortunate, because the notations "$\subseteq, \subset$" here are strongly analogous to the notations $\leq, <$ from arithmetic. One has to take care to know how notation is being used in a given context. (A few authors even use $\subseteq, \subsetneq$ !)

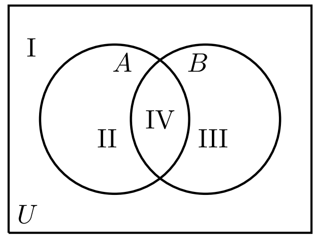



**Figure 1.6:** Most general Venn diagram for two arbitrary sets $A$ and $B$. Here $U$ is some superset of both $A$ and $B$.

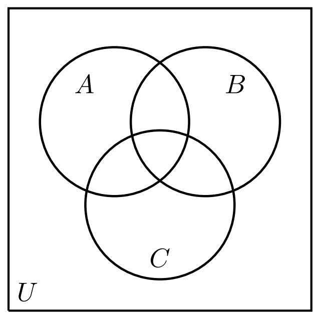



**Figure 1.7:** The most general Venn Diagram for three sets $A$, $B$ and $C$.

**case 1:** $A = B$: all elements of $A$ and $B$ are in Region IV; there are no elements in Regions II and III.

**case 2:** $A \subset B$: there are elements in Regions III and IV, and no elements in Region II.

**case 3:** $B \subset A$: there are elements in Regions II and IV, and no elements in Region III.

**case 4:** $A$ and $B$ share common elements, but neither is a subset of the other: there are elements in Region II, III and IV.

**case 5:** $A$ and $B$ have no common elements: there are no elements in Region IV.

Note that whether or not Region I has elements is irrelevant in the discussion above, though it will become important shortly.

The most general Venn diagram for three sets is given in Figure 1.7, though we will not exhaustively show this to be the most general. It is not important that the sets are represented by circles, but only that there are sufficiently many separate regions and that every case of an element being, or not being, in $A$, $B$ and $C$ is represented. Note that there are three sets for an element to be or not to be a member of, and so there are $2^3 = 8$ subregions needed.

### 1.5.4 Set Operations

When we are given two sets $A$ and $B$, it is natural to combine or compare their memberships with each other and the universe of all elements of interest. In particular, we form new sets called the union and intersection of $A$ and $B$, the difference of $A$ and $B$ (and of $B$ and $A$), and the complement of $A$ (and of $B$). The first three are straightforward, but the fourth requires

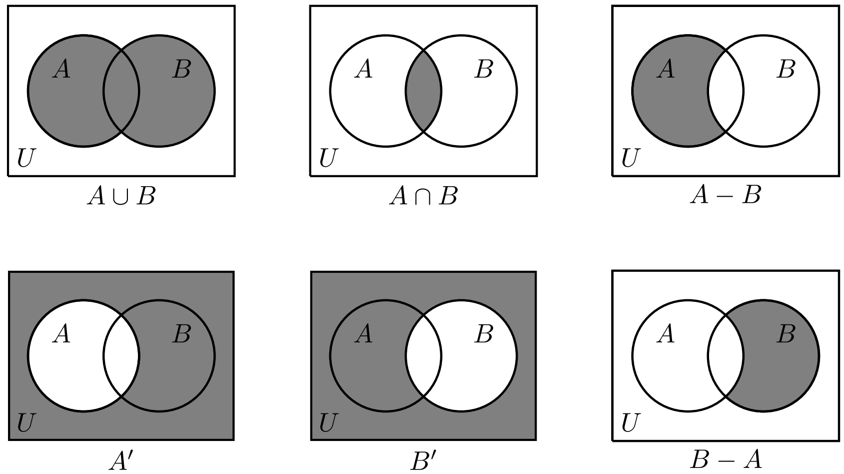



**Figure 1.8:** Some Venn Diagrams involving two sets $A$ and $B$ inside a universal set $U$, which is represented by the whole "box."

some clarification. Usually $A$ and $B$ contain only objects of a certain class like numbers, colors, etc. Thus we take elements of $A$ and $B$ from a specific *universal set* $U$ of objects rather than an all-encompassing universe of all objects. It is unlikely in mathematics that we would need, for instance, to mix numbers with persons and planets and verbs, so we find it convenient to limit our universe $U$ of considered objects. With that in mind (but without presently defining $U$), the notations for these new sets are as follow:

**Definition 1.5.5**

$$A \cup B = \left\{ x \;\middle|\; (x \in A) \vee (x \in B) \right\} \tag{1.90}$$

$$A \cap B = \left\{ x \;\middle|\; (x \in A) \wedge (x \in B) \right\} \tag{1.91}$$

$$A - B = \left\{ x \;\middle|\; (x \in A) \wedge (x \notin B) \right\} \tag{1.92}$$

$$A' = \left\{ x \in U \;\middle|\; (x \notin A) \right\}. \tag{1.93}$$

These are read "$A$ union $B$," "$A$ intersect $B$," "$A$ minus $B$," and "$A$ complement," respectively. Note that in the first three, we could have also written $\left\{ x \in U \middle| \cdots \right\}$, but since $A, B \subseteq U$, there it is unnecessary. Also note that one could define the complement in the following way, though (1.93) is more convenient for symbolic logic computations:

$$A' = \left\{ x \;\middle|\; (x \in U) \wedge (x \notin A) \right\} = U - A. \tag{1.94}$$

These operations are illustrated by the Venn diagrams of Figure 1.8, where we also construct $B'$ and $B - A$. Note the connection between the logical $\vee$ and $\wedge$, and the set-theoretical $\cup$ and $\cap$.[64]

**Example 1.5.1** *Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$ if*

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$
$$B = \{5, 6, 7, 8, 9, 10\}.$$

[64] The set-theoretical "$-$" could be interpreted as "$\wedge \sim \cdots \in$," and if we always assume we know what is the universal set, we can interpret the complement symbol "$'$" as "$\sim \cdots \in$."

*Solution: Though not necessary (and often impossible), we will list these set elements in a table from which we can easily compare the membership.*

$$\begin{array}{ccccccccccccc} A & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & & & \} \ , \\ B & = & \{ & & & & & 5, & 6, & 7, & 8, & 9, & 10 \ \} \ . \end{array}$$

*Now we can compare the memberships using the operations defined earlier.*

$$\begin{aligned} &A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \\ &A \cap B = \{5, 6, 7\}, \\ &A - B = \{1, 2, 3, 4\}, \\ &B - A = \{8, 9, 10\}. \end{aligned}$$

The complements depend upon the identity of the assumed universal set. If in the above example we had $U = \mathbb{N}$, then $A' = \{8, 9, 10, 11, \cdots\}$ and $B' = \{1, 2, 3, 4, 11, 12, 13, 14, 15 \cdots\}$. If instead we took $U = \mathbb{Z}$ we have $A' = \{\cdots, -3, -2, -1, 0, 8, 9, 10, 11, \cdots\}$, for instance. (We leave $B'$ to the interested reader.)

Just as it is important to have a zero element in $\mathbb{R}$ for arithmetic and other purposes, it is also useful in set theory to define a set which contains no elements:

**Definition 1.5.6** *The set with no elements is called the* **empty set**,[65] *denoted* $\varnothing$.

One reason we need such a device is for cases of intersections of disjoint sets. If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7, 8, 9, 10\}$, then $A \cup B = \{1, 2, 3, \cdots, 10\}$, while $A \cap B = \varnothing$. Notice that regardless of the set $A$, we will always have $A - A = \varnothing$, $A - \varnothing = A$, $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$, and $\varnothing \subseteq A$. The last statement is true because, after all, every element of $\varnothing$ is also an element of $A$.[66] Note also that $\varnothing' = U$ and $U' = \varnothing$.

The set operations for two sets $A$ and $B$ can only give us finitely many combinations of the areas enumerated in Figure 1.6. In fact, since each such area is either included or not, there are $2^4 = 16$ different diagram shadings possible for the general case as in Figure 1.6. The situation is more interesting if we have three sets $A$, $B$ and $C$. Using Figure 1.7, we can prove several interesting set equalities. First we have some fairly obvious commutative laws (1.95), (1.96) and associative laws (1.97), (1.98):

$$A \cup B = B \cup A \tag{1.95}$$

$$A \cap B = B \cap A \tag{1.96}$$

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{1.97}$$

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{1.98}$$

Next are the following two *distributive laws*, which are the set-theory analogs to the logical equivalences (1.27) and (1.28), found on page 22.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{1.99}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.100}$$

[65] It is also called the *null set*. Some older texts use empty braces $\varnothing = \{\,\}$.

[66] This is precisely because there are no elements of $\varnothing$; the statement $x \in \varnothing \longrightarrow x \in A$ is vacuously true because $x \in \varnothing$ is false, regardless of $x$.

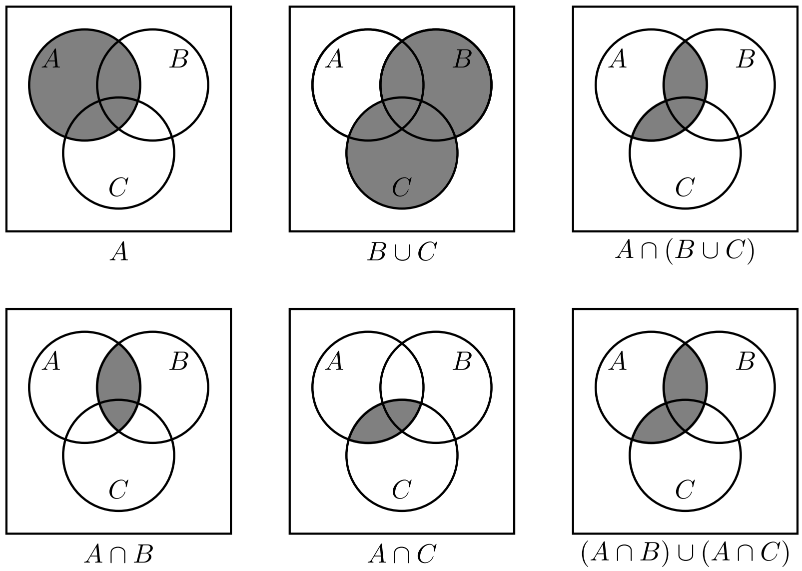



**Figure 1.9:** Venn Diagrams for Example 1.5.2 verifying one of the distributive laws, specifically $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. It is especially important to note how one constructs the third box in each line from the first two.

**Example 1.5.2** *We will show how to prove* (1.99) *using our previous symbolic logic, and then give a visual proof using Venn diagrams. Similar techniques can be used to prove* (1.100). *For the proof that* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*, we use definitions, and* (1.27) *from* page 22 *to get the following:*

$$\begin{aligned} x \in A \cap (B \cup C) &\iff (x \in A) \wedge (x \in B \cup C) \\ &\iff (x \in A) \wedge [(x \in B) \vee (x \in C)] \\ &\iff [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\ &\iff [x \in (A \cap B)] \vee [x \in (A \cap C)] \\ &\iff x \in [(A \cap B) \cup (A \cap C)], \;\; \textit{q.e.d.} \end{aligned}$$

*We proved that* $(\forall x)[(x \in A \cap (B \cup C)) \longleftrightarrow (x \in (A \cap B) \cup (A \cap C))]$*, which is the definition for the sets in question to be equal. The visual demonstration of* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *is given in* Figure 1.9*, where we construct both sets of the equality in stages.*

*To construct the left-hand side of the equation, in the first box we color* $A$*, then* $B \cup C$ *in the second, and finally we take the area from the first, remove the area from the second, and are left with the difference* $A - (B \cup C)$*. To construct the right-hand side of the equation, we color* $A - B$ *and* $A - C$ *in separate boxes. Then we color the intersection of these, which is the area colored in the previous two boxes. This gives us our Venn Diagram for* $(A - B) \cap (A - C)$*. We see that the left- and right-hand sides are the same, and conclude the equality is valid.*

The next two are distributive in nature also:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{1.101}$$

$$A - (B \cap C) = (A - B) \cup (A - C). \tag{1.102}$$

Finally, if we replace $A$ with $U$, we get the set-theoretic version of *de Morgan's Laws:*

$$(B \cup C)' = B' \cap C' \tag{1.103}$$
$$(B \cap C)' = B' \cup C'. \tag{1.104}$$

Note that these are very much like our earlier de Morgan's laws, and indeed use the previous versions (1.3) and (1.4), page 17 (also see page 22) in their proofs. For instance, assuming $x \in U$ where $U$ is fixed, we have

$$\begin{aligned} x \in (B \cup C)' &\iff \sim (x \in B \cup C) \\ &\iff \sim ((x \in B) \vee (x \in C)) \\ &\iff [\sim (x \in B)] \wedge [\sim (x \in C)] \\ &\iff [x \in B'] \wedge [x \in C'] \\ &\iff x \in B' \cap C', \text{ q.e.d.} \end{aligned}$$

That proves (1.103), and (1.104) has a similar proof. It is interesting to prove these using Venn Diagrams as well (see exercises).

**Example 1.5.3** *Another example of how to prove these using logic and Venn diagrams is in order. We will prove* (1.101) *using both methods. First, with symbolic logic:*

$$\begin{aligned} x \in A - (B \cup C) &\iff (x \in A) \wedge [\sim (x \in B \cup C)] \\ &\iff (x \in A) \wedge [\sim ((x \in B) \vee (x \in C))] \\ &\iff (x \in A) \wedge [(\sim (x \in B)) \wedge (\sim (x \in C))] \\ &\iff (x \in A) \wedge (\sim (x \in B)) \wedge (\sim (x \in C)) \\ &\iff (x \in A) \wedge (\sim (x \in B)) \wedge (x \in A) \wedge (\sim (x \in C)) \\ &\iff [(x \in A) \wedge (\sim (x \in B)] \wedge [(x \in A) \wedge (\sim (x \in C))] \\ &\iff (x \in A - B) \wedge (x \in A - C) \\ &\iff x \in (A - B) \cap (A - C), \text{ q.e.d.} \end{aligned}$$

*If we took the steps above in turn, we used the definition of set subtraction, the definition of union,* (1.19)*, associative property of* $\wedge$*, added a redundant* $(x \in A)$*, regrouped, used the definition of set subtraction, and finally the definition of intersection.*

*Now we will see how we can use Venn diagrams to prove* (1.101)*. As before, we will do this by constructing Venn Diagrams for the sets* $A - (B \cup C)$ *and* $(A - B) \cap (A - C)$ *separately, and verify that we get the same sets. We do this in* Figure 1.10*. (If it is not visually clear how we proceed from one diagram to the next "all at once," a careful look at each of the* $2^3 = 8$ *distinct regions can verify the constructions.)*

### 1.5.5 More on Subsets

Before closing this section, a few more remarks should be included on the subject of subsets. Consider for instance the following:

**Example 1.5.4** *Let* $A = \{1, 2\}$*. List all subsets of* $A$*.*

*Solution: As* $A = \{1, 2\}$ *has two elements, it can have subsets which contain zero elements, one element, or two elements. The subsets are thus* $\varnothing$*,* $\{1\}$*,* $\{2\}$ *and* $\{1, 2\} = A$*.*

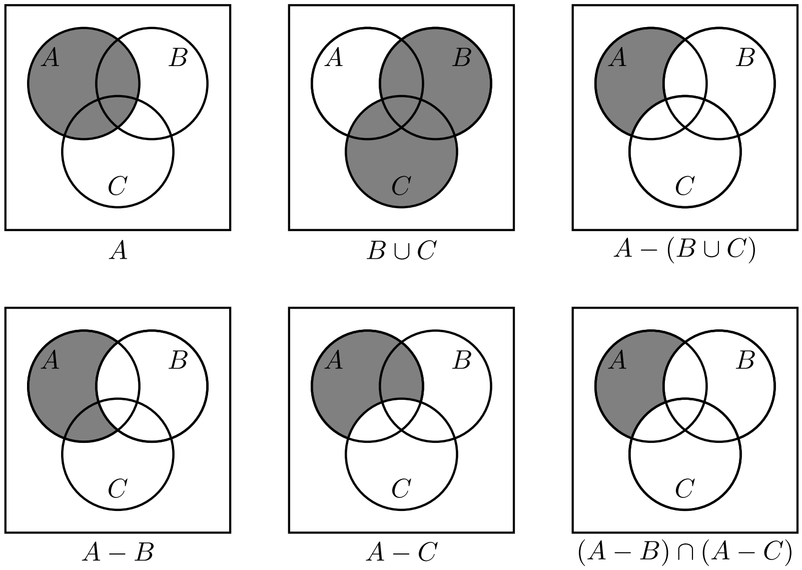



**Figure 1.10:** Venn Diagrams for Example 1.5.3 verifying that $A-(B\cup C) = (A-B)\cap(A-C)$.

It is common for novices studying sets to forget that $\varnothing \subseteq A$, and $A \subseteq A$, though by definition,

$$\begin{aligned} x \in \varnothing &\implies x \in A \qquad \text{(vacuously)},\\ x \in A &\implies x \in A \qquad \text{(trivially)}. \end{aligned}$$

If one wanted only *proper* subsets of $A$, those would be $\varnothing, \{1\}, \{2\}$ (we omit the set $A$).

Note that with our set $A = \{1, 2\}$, we can reduce rephrase the question of which subset we might refer to, instead into a question of exactly which elements are in it, from the choices 1 and 2. In other words, given a subset $B \subseteq A$, which (if any) of the following are true: $1 \in B$, $2 \in B$. From these statements we can construct a truth table-like structure to describe every possible subset of $A$:

$A = \{1, 2\}$

| $1 \in B$ | $2 \in B$ | subset $B$ |
|---|---|---|
| T | T | $\{1, 2\} = A$ |
| T | F | $\{1\}$ |
| F | T | $\{2\}$ |
| F | F | $\varnothing$ |

Similarly, a question about subsets $B$ of $A = \{a, b, c\}$ can be placed in context of a truth table-like construct:

$A = \{a, b, c\}$

| $a \in B$ | $b \in B$ | $c \in B$ | subset $B$ |
|---|---|---|---|
| T | T | T | $\{a, b, c\} = A$ |
| T | T | F | $\{a, b\}$ |
| T | F | T | $\{a, c\}$ |
| T | F | F | $\{a\}$ |
| F | T | T | $\{b, c\}$ |
| F | T | F | $\{b\}$ |
| F | F | T | $\{c\}$ |
| F | F | F | $\varnothing$ |

It would not be too difficult to list the elements of $A = \{1, 2, 3\}$ by listing subsets with zero, one, two and three elements separately, i.e., $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, but if we were to need to list subsets of a set with significantly more elements, it might be easier to use the lexicographical order embedded in the truth table format to exhaust all the possibilities. The only disadvantage is that the order in which subsets are listed might not be quite as natural as the order we would likely find if we listed subsets with zero, one, two elements and so on.

# Exercises

1. Draw all the $2^4 = 16$ possible shadings for Figure 1.6, page 60. Then use the sets $A$, $B$ and $U$, together with unions, intersections, complements and set differences ($\cup, \cap, ', -$) to write a corresponding expression for the each of the shaded areas. Note that Figure 1.8 illustrates six of them. Also note that there may be more than one way of representing a set. For example, $A' = U - A$.

2. Use symbolic logic **and** Venn Diagrams (as in Examples 1.5.2 and 1.5.3) to prove the other set equalities:

   (a) (1.97): $A \cup (B \cup C) = (A \cup B) \cup C$

   (b) (1.98): $A \cap (B \cap C) = (A \cap B) \cap C$

   (c) (1.102): $A - (B \cap C) = (A - B) \cup (A - C)$

   (d) (1.99): $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (e) (1.103): $(B \cup C)' = B' \cap C'$

   (f) (1.104): $(B \cap C)' = B' \cup C'$

3. Use Venn Diagrams to draw and determine a simpler way of writing the following sets:

   (a) $A - (B - A) =$

   (b) $A - (A - B) =$

   (c) $(A - B) \cap (B - A) =$

4. Answer each of the following.

   (a) If $A \subseteq B$, what is $A - B$?

   (b) If $A \subset B$, what can you say about $B - A$?

   (c) Referring to Figure 1.8, what are $U'$ and $\varnothing'$?

   (d) Suppose $A \subseteq B$. How are $A'$ and $B'$ related?

   (e) Suppose $A - B = A$. What is $A \cap B$?

   (f) Suppose $A - B = B$. What is $B$? What is $A$?

   (g) Is it possible that $A \subseteq B$ and $B \subseteq A$?

5. The set of *irrational numbers* is the set

$$\mathbb{I} = \left\{ x \in \mathbb{R} \;\middle|\; x \text{ is not rational} \right\}.$$

   Using previously-defined sets and set notation, find a concise definition of $\mathbb{I}$.

6. Many books define the *symmetric difference* between two sets $A$ and $B$ by

$$A \vartriangle B = (A - B) \cup (B - A). \tag{1.105}$$

   (a) Use a Venn Diagram to show that $A \vartriangle B = (A \cup B) - (A \cap B)$.

   (b) Is it true that $A \vartriangle B = B \vartriangle A$?

   (c) Use a Venn Diagram to show that $A \vartriangle (B \vartriangle C) = (A \vartriangle B) \vartriangle C$.

   (d) Calculate $A \vartriangle A$, $A \vartriangle U$, and $A \vartriangle \varnothing$.

   (e) If $A \subseteq B$, what is $A \vartriangle B$? What is $B \vartriangle A$?

   (f) Define $A \vartriangle B$ in a manner similar to the definitions (1.90)–(1.93). That is, replace the dots with a description of $x$ in the following:

$$A \vartriangle B = \left\{ x \;\middle|\; \cdots \right\}.$$

7. Redraw the Venn Diagram of Figure 1.7 and label each of the eight disjoint areas I–VIII. Then use the sets $U$, $A$, $B$ and $C$, together with the operations $\cup$, $\cap$, $-$ and $'$, to find a definition of each of these sets I–VIII.

8. How many different shading combinations are there for the general Venn Diagram for 3 sets $A$, $B$ and $C$? (See Figure 1.7.) Speculate about how many combinations there are for 4 sets, 5 sets, and $n$ sets. Test your hypothesis for $n = 0$ and $n = 1$.

9. Show that $A \subseteq B \iff B' \subseteq A'$. (See the discussion immediately following (1.86), page 55.)

10. A useful concept in set theory is *cardinality* of a set $S$, which we denote $n(S)$, which can be defined to be the number of elements in a set if the set is finite. Thus $n(\{1,2,3,8,9,10\}) = 6$.

    (a) Use a Venn Diagram to show that

    $$n(S \cup T) = n(S) + n(T) - n(S \cap T). \tag{1.106}$$

    (b) Show that if $n(S \cup T) = n(S) + n(T)$, then $S \cap T = \varnothing$.

11. Given

    $$\begin{aligned} U &= \{1,2,3,\cdots,12\}, \\ A &= \{1,2,3,4,5,6\}, \\ B &= \{2,3,4,6,7,8\}, \\ C &= \{3,5,7\}, \\ D &= \{3,5,7,11\}, \end{aligned}$$

    find each of the following:

    (a) $(A - B) \cup (B - A)$,

    (b) $(A - D)'$,

    (c) the number of subsets $C$,

    (d) the number of subsets of $D$.

    (e) Use the results from (c) and (d) to determine the number of subsets that a set of 5 elements should have.

## 1.6 Epilogue: Proofs Without Truth Tables

Truth tables are the most exhaustive tools for studying compound statements, because they simply check every case. This makes them useful tools to fall back upon if necessary, but as we learn and memorize more equivalences it is useful to put what we already showed to work in other proofs. For instance, recall the following equivalences, here written with our new notation:

1. $P \longrightarrow Q \iff (\sim P) \vee Q$,
2. $\sim [P \vee Q] \iff (\sim P) \wedge (\sim Q)$,
3. $\sim (\sim P) \iff P$.

We will use these principles, in exactly the order above, to produce a very concise proof in the following example.

**Example 1.6.1** *Prove without truth tables that* $\sim (P \longrightarrow Q) \iff P \wedge (\sim Q)$.
*Solution:*

$$\begin{aligned} \sim (P \longrightarrow Q) &\iff \sim [(\sim P) \vee Q] \\ &\iff [\sim (\sim P)] \wedge (\sim Q) \\ &\iff P \wedge (\sim Q), \qquad \textit{q.e.d.} \end{aligned}$$

To show a statement is a tautology it is enough to show that it is logically equivalent to $\mathcal{T}$. We can use this fact to prove valid logical equivalences (i.e., show $\longleftrightarrow$ gives a tautology) and valid logical implications (i.e., show $\longrightarrow$ gives a tautology). There are often easier ways to prove valid equivalences, but for valid implications this method can be useful, as in the example below.

**Example 1.6.2** *We revisit* Example 1.3.2, *proving* (1.47), page 29 *using this notation. That is, we wish to show that* $(P \wedge Q) \implies P$ *by showing*

$$(P \wedge Q) \longrightarrow P \iff \mathcal{T}.$$

*The proof will run as follows:*

$$\begin{aligned} (P \wedge Q) \longrightarrow P &\iff \sim (P \wedge Q) \vee P \\ &\iff (\sim P) \vee (\sim Q) \vee P \\ &\iff (\sim P) \vee P \vee (\sim Q) \\ &\iff \mathcal{T} \vee (\sim Q) \\ &\iff \mathcal{T}, \text{ q.e.d.} \end{aligned}$$

This kind of proof requires much practice, but also can be very satisfying when completely mastered. The highly interested student can work all of the exercises, and perhaps continue by attempting, without truth tables, proofs of the remaining valid equivalences and implications listed in the next subsection. We leave a couple of more difficult examples for later in the next (and final) subsection.

**Example 1.6.3** *Here we expand the simple statement* $P \longleftrightarrow Q$ *using, respectively, the obvious equivalent statement, followed by* (1.29), page 22, *several applications of the distributive rule*

(1.27), page 22 *and finally* (1.39), page 26.[67]

$$\begin{aligned}
P \longleftrightarrow Q &\iff (P \longrightarrow Q) \wedge (Q \longrightarrow P) \\
&\iff [(\sim P) \vee Q] \wedge [(\sim Q) \vee P] \\
&\iff \Big\{[(\sim P) \vee Q] \wedge (\sim Q)\Big\} \bigvee \Big\{[(\sim P) \vee Q] \wedge P\Big\} \\
&\iff \Big\{[(\sim P) \wedge (\sim Q)] \vee [Q \wedge (\sim Q)]\Big\} \bigvee \Big\{[(\sim P) \wedge P] \vee [Q \wedge P)]\Big\} \\
&\iff \Big\{[(\sim P) \wedge (\sim Q)] \vee \mathcal{F}\Big\} \bigvee \Big\{\mathcal{F} \vee [Q \wedge P)]\Big\} \\
&\iff \{(\sim P) \wedge (\sim Q)\} \vee \{P \wedge Q\}.
\end{aligned}$$

*Though we had a good understanding of* $P \longleftrightarrow Q$ *before—that it is true if and only if* $P$ *and* $Q$ *are both true or both false—we might not have been as quick to declare obvious what we just derived when read backwards:*

$$(P \wedge Q) \vee ((\sim P) \wedge (\sim Q)) \iff P \longleftrightarrow Q. \tag{1.107}$$

**Example 1.6.4 (A Rather Sophisticated Proof)** *We will prove* (1.52), page 32 *using its predecessors, particularly* (1.49) *and the distributive laws. The proof is not as straightforward as using truth tables, but it is worthwhile to follow along.*

$$\begin{aligned}
(P \longrightarrow Q) \wedge (Q \longrightarrow R) &\iff [(\sim P) \vee Q] \wedge [(\sim Q) \vee R] \\
&\iff \Big\{[(\sim P) \vee Q] \wedge (\sim Q))\Big\} \vee \Big\{[(\sim P) \vee Q] \wedge R\Big\} \\
&\iff \Big\{[(\sim P) \wedge (\sim Q)] \vee [Q \wedge (\sim Q]\Big\} \vee \Big\{[(\sim P) \vee Q] \wedge R\Big\} \\
&\iff \Big\{[(\sim P) \wedge (\sim Q)] \vee \mathcal{F}\Big\} \vee \Big\{[(\sim P) \vee Q] \wedge R\Big\} \\
&\iff \Big\{\underbrace{(\sim P) \wedge (\sim Q)}_{A}\Big\} \vee \Big\{\underbrace{[(\sim P) \vee Q]}_{B} \wedge \underbrace{R}_{C}\Big\} \\
&\iff \Big\{\underbrace{[(\sim P) \wedge (\sim Q)]}_{A} \vee \underbrace{[(\sim P) \vee Q]}_{B}\Big\} \wedge \Big\{\underbrace{[(\sim P) \wedge (\sim Q)]}_{A} \vee \underbrace{R}_{C}\Big\} \\
&\iff (A \vee B) \wedge \Big\{\Big((\sim P) \vee R\Big) \wedge \Big((\sim Q) \vee R\Big)\Big\} \\
&\iff \Big\{\underbrace{\Big(A \vee B\Big) \vee \Big((\sim Q) \vee R\Big)}_{D}\Big\} \wedge \Big\{(\sim P) \vee R\Big\} \\
&\iff \Big\{\underbrace{\Big(A \vee B\Big) \vee \Big((\sim Q) \vee R\Big)}_{D}\Big\} \wedge \Big\{P \longrightarrow R\Big\} \\
&\implies P \longrightarrow R, \ \textit{q.e.d.}
\end{aligned}$$

*Here we used some labels and substitutions* $A, B, C, D$ *for clarity and space considerations. The actual form of* $D$ *did not matter. Our final step was of the form* $D \wedge (P \longrightarrow R) \implies P \longrightarrow R$, *i.e.,* (1.49), page 32. *Note also that it is enough to have* $\implies$ *all the way down in such a proof,*

[67] We include the references to previous valid equivalences and implications in case they are needed, but the reader should attempt to read the proof first without resorting to the references. Indeed the reader might not feel the need to look up the previous results at all, if comfortable with each step.

*since we could then follow the "arrows" from the first statement through to the last. However it is best to work with equivalences as long as possible, to avoid losing too much information in early steps where the goal is not so easily concluded.*

Clearly this can be a messy process. Fortunately, we rarely need to expand things out so far. What is important is that we now have a language of logic and some very general valid equivalences and implications which

1. we can prove using truth tables—a process with an obvious road map—if necessary;
2. will become more and more natural for us to see as valid by inspection, and be prepared to use;
3. and (later) will make shorter, clearer work of our mathematical arguments.

Some further observations and insights are introduced in the exercises. Many are very important and will be referred to throughout the text.

# Exercises

1. Show without truth tables the following. (Hint: There is a very easy way and a very long way.)

$$(P \longrightarrow Q) \wedge ((\sim P) \longrightarrow (\sim Q)) \iff P \longleftrightarrow Q. \quad (1.108)$$

Does this make sense? (You should relate this to English statements of these forms.)