

Chapter 6

Basic Integration

In this chapter we will consider the problem of recovering a function from knowledge of its derivative, or, equivalently, for a given function we will try to find another function whose derivative is the given function's. The general process is called *antidifferentiation*, or *integration*. The meaning of the first term is obvious: we are working backwards from the derivative to the function. The meaning of the other name for the process is mainly left for Chapter 8.

The main purpose of this chapter is to develop the first, basic techniques for computing antiderivatives $F(x)$ for a given function $f(x)$, i.e., given $f(x)$ we look for $F(x)$ so that

$$F'(x) = f(x). \quad (6.1)$$

As we will see early in this chapter and throughout the next, antidifferentiation (finding some such $F(x)$), a.k.a. integration, is less straightforward than differentiation (finding $f'(x)$). However, there are easily as many applications of antidifferentiation as there are of differentiation so it is a worthwhile process. In the first section we will limit ourselves to two applications:

1. Given the slope $f'(x)$, find the function $f(x)$ by antidifferentiation;¹
Moreover, given $f''(x)$, find $f'(x)$, and then $f(x)$.
2. Given velocity v , find position s .
Moreover, given acceleration a , find v and then s .

Later in the chapter we will look at the geometric significance of antiderivatives $F(x)$ of a function $f(x)$. Just as the geometric meaning of slope gave us a useful perspective for arriving at derivative theorems (mean value theorem, first derivative test, etc.), so too will the antiderivatives benefit from geometric analysis. To make that analysis will require us to consider another major theoretical device, namely *Riemann sums*, which—together with the Fundamental Theorem of Calculus—will open the topic of integration to innumerable applications. To illustrate the reasonableness of the Fundamental Theorem of Calculus, we will again look closely at the velocity-position connection as well, in Chapter 8.

In Chapter 7 we will develop more advanced techniques of antidifferentiation, so that we can use all of our integration techniques in Chapter 8, which is devoted to applications.

For now we will concentrate on the actual computation of antiderivatives of the more basic types. In the first section we will limit ourselves to those which arise from our known derivative formulas. In subsequent sections we will then explore the substitution technique, which is the antidifferentiation analog to the chain rule and is thus arguably the most important of the integration techniques. It will be developed at length.

¹In this application, the part of $f(x)$ in (6.1) is played by $f'(x)$, while the part of $F(x)$ is played by $f(x)$.

6.1 First Indefinite Integrals (Antiderivatives)

In this section we introduce *antiderivatives*, which are exactly what the name implies. These are also called *indefinite integrals* for reasons which will eventually become clear.

6.1.1 Indefinite Integrals and Constants of Integration

Definition 6.1.1 Consider a function $f(x)$ which is defined on an open interval (a, b) . Another function $F(x)$, also defined on (a, b) is called an **antiderivative** of $f(x)$ on the same interval if and only if $F'(x) = f(x)$ on (a, b) .

If instead $f(x)$ is defined on a closed interval $[a, b]$, we still call $F : [a, b] \rightarrow \mathbb{R}$ an **antiderivative** of $f(x)$ on $[a, b]$ if and only if

$$\begin{aligned} F'(x) &= f(x), & x &\in (a, b), \\ \lim_{\Delta x \rightarrow 0^+} \frac{F(a + \Delta x) - F(a)}{\Delta x} &= f(a), \\ \lim_{\Delta x \rightarrow 0^-} \frac{F(b + \Delta x) - F(b)}{\Delta x} &= f(b). \end{aligned}$$

In other words, on an open interval we require $F'(x) = f(x)$, while on the closed interval we also require the right derivative of F to be f at $x = a$, and the left derivative of F to be f at $x = b$.

Notice that the definition implies that $F(x)$ is continuous on the interval in question (since $F' = f$ exists implies continuity of F). For a simple example, consider $f(x) = 2x + 3$ on any open (or nontrivial closed) interval. An antiderivative of $f(x)$ can be $F(x) = x^2 + 3x$, since then $F'(x) = 2x + 3 = f(x)$. However, another perfectly good antiderivative can be $F(x) = x^2 + 3x + 5$, or $F(x) = x^2 + 3x - 100,000$, since the derivative of the trailing constant term will always be zero. In logical terms we can write

$$\begin{aligned} F(x) = x^2 + 3x &\implies F'(x) = 2x + 3, \\ F(x) = x^2 + 3x + 5 &\implies F'(x) = 2x + 3, \\ F(x) = x^2 + 3x - 100,000 &\implies F'(x) = 2x + 3, \\ F(x) = x^2 + 3x + C, \text{ some } C \in \mathbb{R} &\iff F'(x) = 2x + 3. \end{aligned} \tag{6.2}$$

That the last one (6.2) is an equivalence we will prove shortly. To signify that equivalence, we write

$$\int (2x + 3) dx = x^2 + 3x + C, \quad C \in \mathbb{R}. \tag{6.3}$$

We call the right hand side of (6.3) *the most general antiderivative*, or just *the antiderivative*, of $2x + 3$ (with respect to x). It is also called the *indefinite integral* of $2x + 3$ (again with respect to x), and we will eventually migrate to using that term as our default.² The constant C is called the *constant of integration*, since it must be included to achieve all solutions to the question of what is an antiderivative of $f(x) = 2x + 3$. On the left hand side of (6.3) we have

- \int : the *integral sign*, derived from a old style German S for reasons we will discuss later;

²The indefinite integral has a strong connection to the very important *definite integral*, which is a measure of accumulated change as computed from the instantaneous rate of change. This computation is our eventual goal and—to restate the introduction to this chapter—the connection between antiderivatives (indefinite integrals) and accumulated change (definite integrals) is precisely the subject of the Fundamental Theorem of Calculus.

- $(2x + 3)$: the *integrand*, whose antiderivatives we seek; and
- dx : the differential of x , which signifies which variable the antiderivative is with respect to.

We now note that, indeed, all antiderivates of $2x + 3$ are necessarily of the form $x^2 + 3x + C$. To prove this, on some interval define $F(x) = x^2 + 3x$, which we can easily see is an antiderivative of $2x + 3$ on that interval. Next suppose $G(x)$ is another such antiderivative, i.e., that $G'(x) = 2x + 3$, on that same interval. Then on that interval

$$\frac{d}{dx}[G(x) - F(x)] = G'(x) - F'(x) = (2x + 3) - (2x + 3) = 0 \implies G(x) - F(x) = C,$$

for some $C \in \mathbb{R}$. Thus any antiderivative $G(x)$ must be of the form $G(x) = F(x) + C$, q.e.d.³ To be clear on the notation, we now insert the following definition.

Definition 6.1.2 *If $F(x)$ is an antiderivative of $f(x)$, with respect to x , on the interval I , then on that interval we write*

$$\int f(x) dx = F(x) + C, \quad (6.4)$$

where C is an arbitrary constant of integration.

Example 6.1.1 *Consider $f(x) = 2 \sin x \cos x$. One antiderivative is $F(x) = \sin^2 x$, since*

$$F'(x) = \frac{d \sin^2 x}{dx} = \frac{d(\sin x)^2}{dx} = 2 \sin x \cdot \frac{d \sin x}{dx} = 2 \sin x \cos x.$$

However, another antiderivative is $G(x) = -\cos^2 x$, since

$$G'(x) = \frac{d(-\cos^2 x)}{dx} = -\frac{d(\cos x)^2}{dx} = -\left[2 \cos x \cdot \frac{d \cos x}{dx}\right] = -[2 \cos x(-\sin x)] = 2 \sin x \cos x.$$

Note that

$$F(x) - G(x) = \sin^2 x - (-\cos^2 x) = \sin^2 x + \cos^2 x = 1,$$

so we see that F and G do actually differ by a constant. To report the most general antiderivative of $f(x) = 2 \sin x \cos x$, either of the following are valid (but understood to have different “ C ’s”):

$$\begin{aligned} \int 2 \sin x \cos x dx &= \sin^2 x + C, \\ \int 2 \sin x \cos x dx &= -\cos^2 x + C. \end{aligned}$$

Especially when dealing with trigonometric functions—and all their interconnectedness by various identities—it is common to find very different-looking forms of the general antiderivative, all of which differ by constants from each other. It is occasionally important to be alert for apparent discrepancies which are explained by this nature of the general antiderivative.

³Recall that a function with the zero function for its derivative on an interval must be constant on that interval:

$$(\forall x \in I)[(F(x) - G(x))' = 0] \implies (\exists C \in \mathbb{R})(\forall x \in I)[F(x) - G(x) = C].$$

Example 6.1.2 Suppose $f(x) = x + 1$. Then both forms below are general antiderivatives:

$$\int (x + 1) dx = \frac{1}{2}x^2 + x + C,$$

$$\int (x + 1) dx = \frac{1}{2}(x + 1)^2 + C.$$

We can see this by taking derivatives of each. We can also see that if we label $F(x) = \frac{1}{2}x^2 + x$ and $G(x) = \frac{1}{2}(x + 1)^2$, so the antiderivatives above are just $F(x)$ and $G(x)$ plus constants, respectively, and then compute

$$F(x) - G(x) = \left[\frac{1}{2}x^2 + x \right] - \left[\frac{1}{2}(x^2 + 2x + 1) \right] = -\frac{1}{2},$$

so these do differ by a constant, as expected.

6.1.2 Power Rule for Integrals

Where the rules for computing derivatives were straightforward (which is not to say necessarily “easy”), those for computing antiderivatives are not so algorithmic. Indeed the methods are varied. Nonetheless, they are necessary to learn for a reasonably complete understanding of standard calculus, and we begin with the *power rule for integrals*:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1, \quad (6.5)$$

$$\int \frac{1}{x} dx = \ln|x| + C. \quad (6.6)$$

Here the intervals in question are those upon which x^n is defined. To check (6.6), we simply notice $\frac{d}{dx} \ln|x| = \frac{1}{x}$. For (6.5) we compute:

$$\frac{d}{dx} \left[\frac{1}{n+1}x^{n+1} \right] = \frac{1}{n+1} \cdot (n+1)x^{[(n+1)-1]} = x^n, \quad \text{q.e.d.}$$

In performing the above computation, we must notice that $n + 1$ and $1/(n + 1)$ are constants, and so are preserved throughout the computation (and do not, for instance, require product/quotient/chain rules since they do not vary). We also note that the formula to the right of (6.5) is meaningless for the case $n = -1$, so we need (6.6) for that case.

When checking any antiderivative by differentiation, it is customary to not include the arbitrary additive constant, since its derivative is zero. However, it is certainly correct to include it, as in $\frac{d}{dx} [\ln|x| + C] = \frac{1}{x} + 0 = \frac{1}{x}$.

We can apply the power rule immediately as in the following:

$$\int x^2 dx = \frac{1}{3}x^3 + C,$$

$$\int x^3 dx = \frac{1}{4}x^4 + C,$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1}x^{-1} + C = -x^{-1} + C = \frac{-1}{x} + C,$$

$$\int \frac{x}{x^2} dx = \int \frac{1}{x} dx = \ln|x| + C.$$

We can check all of these by taking derivatives of our answers, with respect to x (i.e., by applying d/dx).⁴ As with derivatives, many functions which are powers of the variable are not explicitly written as such. Furthermore as with derivatives, the variable name in antiderivative formulas does not matter as long as it is matched in the differential:

$$\begin{aligned}\int t^9 dt &= \frac{1}{10}t^{10} + C, \\ \int \sqrt{u} du &= \int u^{1/2} du = \frac{1}{\frac{1}{2}+1}u^{[\frac{1}{2}+1]} + C = \frac{2}{3}u^{3/2} + C, \\ \int \frac{1}{z\sqrt[4]{z}} dz &= \int z^{-5/4} dz = \frac{1}{\frac{-5}{4}+1}z^{[\frac{-5}{4}+1]} + C = \frac{z^{-1/4}}{-\frac{1}{4}} + C = \frac{-4}{\sqrt[4]{z}} + C.\end{aligned}$$

To check these, we can apply respectively d/dt , d/du and d/dz . Before we go on we note that (6.5), interpreted formally, applies to the case $n = 0$ as well: $\int x^0 dx = \frac{1}{0+1}x^{0+1} + C$, i.e.,

$$\int 1 dx = x + C. \quad (6.7)$$

Taking the derivative of the right-hand side of (6.7) quickly shows it is in fact true. This integral is often abbreviated

$$\int dx = x + C. \quad (6.8)$$

As with the chain rule computations, the differential dx is formally treated as a factor as in the following:

$$\int \frac{dx}{x^3} = \int x^{-3} dx = \frac{1}{-2}x^{-2} + C = \frac{-1}{2x^2} + C.$$

Later we will see this treatment of dx justified and exploited in several contexts.

Now we state two very general results which may seem obvious, but are worth exploring with some care because of a technical consideration regarding the constant of integration.⁵ These are also useful to us immediately because they allow to use the power rule multiple times to compute the derivatives of polynomials.

Theorem 6.1.1 *Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$ on some interval in consideration, and that $k \in \mathbb{R}$ is a fixed constant. Then on that same interval we have*

$$\int [f(x) + g(x)] dx = F(x) + G(x) + C, \quad (6.9)$$

$$\int [k \cdot f(x)] dx = kF(x) + C. \quad (6.10)$$

where $C \in \mathbb{R}$ is a constant of integration.

⁴In fact, it is a very useful exercise to check these antiderivatives by quick mental calculations. Since the derivative formulas have been used extensively to this point, the processes of computing antiderivatives can be well-informed by their connections to known derivative techniques. In particular, mistakes in computing antiderivatives can often be immediately corrected. Perhaps even more importantly, the approximate form of an antiderivative can often be anticipated. For instance, we know that the derivative of a fourth-degree polynomial is necessarily a third-degree polynomial. It should be clear (though some argument is necessary to prove) that the antiderivative of a third-degree polynomial is necessarily a fourth-degree polynomial. Anticipating the final form is also very useful in substitution problems, which are introduced in Section 6.4 and ubiquitous thereafter.

⁵Many calculus students become careless about this constant of integration and just “tack it on the end” when computing antiderivatives. However, its correct placement is crucial in several contexts, so it is useful to be vigilant from the beginning of integration study. Carelessness in its placement can cause trouble already in this section, but will be particularly troublesome in subsequent third-semester calculus and in differential equations studies.

These can be proved by taking derivatives. For instance, $\frac{d}{dx}[kF(x)] = k \cdot \frac{d}{dx}F(x) = kF'(x) = kf'(x)$. Note that this theorem can be rewritten

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx, \quad (6.11)$$

$$\int kf(x) dx = k \int f(x) dx, \quad k \neq 0, \quad (6.12)$$

$$\int 0 dx = C. \quad (6.13)$$

A moment's reflection shows that we do need (6.12) and (6.13) to catch both cases summarized in (6.10), else we lose the constant of integration if we let $k = 0$ in (6.12). (Note that, indeed, $\frac{d}{dx}C = 0$ which verifies (6.13).) More importantly, these new forms (6.11)–(6.13) are not inconsistent with those of the theorem when we consider the arbitrary constants. For instance, if we assume $F'(x) = f(x)$ and $G'(x) = g(x)$ as in the theorem, then we can write (6.11) as follows:

$$\begin{aligned} \int f(x) dx + \int g(x) dx &= (F(x) + C_1) + (G(x) + C_2) \\ &= F(x) + G(x) + \underbrace{(C_1 + C_2)}_{\text{"C"}} = F(x) + G(x) + C, \end{aligned}$$

where C_1 and C_2 are arbitrary constants, and their sum will also be an arbitrary constant which we can name C .

With the above theorem and the power rule, we can now compute the indefinite integrals of polynomials and other linear combinations of powers.

Example 6.1.3 Consider the following integrals:

$$(a) \int (4x^2 - 9x + 7) dx = 4 \cdot \frac{x^3}{3} - 9 \frac{x^2}{2} + 7 \cdot x + C = \frac{4}{3}x^3 - \frac{9}{2}x^2 + 7x + C,$$

$$(b) \int \left(\frac{3x-9}{x^3} \right) dx = \int (3x^{-2} - 9x^{-3}) dx = -3 \cdot \frac{x^{-1}}{-1} - 9 \cdot \frac{x^{-2}}{-2} + C = \frac{3}{x} + \frac{9}{2x^2} + C,$$

$$(c) \int (5x^2)^3 dx = \int 125x^6 dx = 125 \cdot \frac{x^7}{7} + C,$$

$$(d) \int (x^2 + 7)^2 dx = \int (x^4 + 14x^2 + 49) dx = \frac{x^5}{5} + 14 \cdot \frac{x^3}{3} + 49x + C,$$

$$(e) \int \sqrt{2x} dx = \int \sqrt{2} \sqrt{x} dx = \int 2^{1/2} x^{1/2} dx = 2^{1/2} \cdot \frac{x^{3/2}}{3/2} + C = \sqrt{2} \cdot \frac{2}{3} x^{3/2} + C.$$

When we use an integration rule such as the power rule for integrals, as with derivatives it is important that the variable which the antiderivative is with respect to matches the variable in the function. For instance,

$$\int x^3 dx = \frac{1}{4}x^4 + C, \quad (6.14)$$

$$\int w^3 dw = \frac{1}{4}w^3 + C, \quad (6.15)$$

$$\int (5x - 11)^3 dx \neq \frac{1}{4}(5x - 11)^4 + C. \quad (6.16)$$

What goes wrong in (6.16) is the integral analog to what goes wrong below (which is that we need the chain rule to make the variables of differentiation match):

$$\begin{aligned}\frac{d}{dx} [(5x - 11)^4] &\neq 4(5x - 1)^3; \\ \frac{d}{dx} [(5x - 11)^4] &= 4(5x - 11)^3 \cdot \frac{d(5x - 11)}{dx} = 4(5x - 11)^3 \cdot 5 \neq 4(5x - 11)^3,\end{aligned}$$

The problem is that the differential, dx , is that of x and not $(5x - 11)$. In the next section we will address a kind of integral version of the chain rule (commonly known as *integration by substitution* for reasons which will be clear later), which would make short work of this integral. Without it, we may need to expand $(5x - 11)^3$ as a polynomial, or guess the solution and check that it works, and possibly make adjustments. Either way, it should suffice to point out that we need to take care in using integral formulas such as the integral power rule, page 402.⁶

6.1.3 Finding C

Many times we are interested in a particular antiderivative. Since all antiderivatives (on a particular interval) differ by a constant, this means we need to find one antiderivative, and then try to “fix,” i.e., determine, the constant.

Example 6.1.4 Find $f(x)$ so that $f'(x) = 2x$ and $f(3) = 7$.

Solution: For a problem such as this, it is common to write

$$f(x) = \int f'(x) dx,$$

where it is understood that we will eventually find the exact antiderivative so that the function is well-defined. For our particular problem, one might continue to write

$$f(x) = \int 2x dx = 2 \cdot \frac{x^2}{2} + C = x^2 + C.$$

Now we find the particular C , and we do this by inputting the “datum” (sometimes called “data point”) $f(3) = 7$. Graphically this means that the point $(3, 7)$ is on the curve. Since $f(x) = x^2 + C$, we find C using this datum:

$$\begin{aligned} & f(3) = 7 \\ \iff & 3^2 + C = 7 \\ \iff & 9 + C = 7 \\ \iff & C = -2. \end{aligned}$$

Thus $f(x) = x^2 - 2$.

⁶Without giving away the substitution technique, we will note here that we can rewrite the integral in (6.16) so the differential matches the term $(5x - 11)$. The argument would be the analog of our early chain rule expansions, such as (4.28), page 288. The idea is that $d(5x - 11) = 5dx$ (recall the meaning of $d(5x - 11)/dx$), and so $dx = d(5x - 11)/5$, which allows us to rewrite the integral as follows:

$$\int (5x - 11)^3 dx = \int (5x - 11)^3 \frac{d(5x - 11)}{5} = \frac{1}{5} \cdot \frac{1}{4} \cdot (5x - 11)^4 + C = \frac{1}{20}(5x - 11)^4 + C.$$

Except for the factor of $\frac{1}{5}$ in the second interval, that rewriting had an integral power rule form.

Our integration by substitution method will be more systematic than the above computation. Consequently it will read better, and be less error-prone. That method will then be called upon extensively from then on.

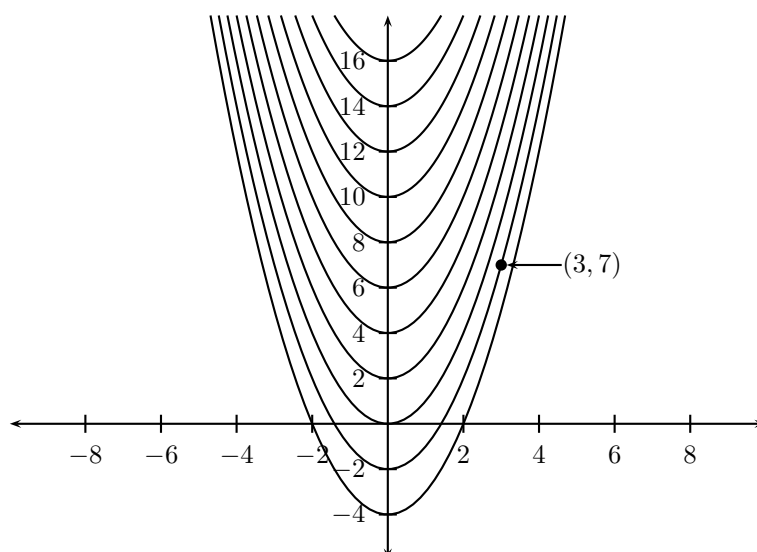


Figure 6.1: Partial view of the family of curves $y = x^2 + C$. For Example 6.1.4, we needed to find the value of C satisfying $f'(x) = 2x$, i.e., $f(x) = x^2 + C$, so that $(3, 7)$ was on the curve. Note that the slopes of all curves given above are the same for a given x -value, but only one passes through $(3, 7)$.

A graphical way of interpreting the example above is to realize that all the curves $y = x^2 + C$ are parabolas, and in fact are just vertical shifts of the curve $y = x^2$. Our task in Example 6.1.4 was then to find which shift satisfies both $f'(x) = 2x$ and $f(3) = 7$. In Figure 6.1, $y = x^2 + C$ is graphed for various values of C . Once we require the graph to pass through a particular point—in this case the point $(3, 7)$, we “pin down” a particular curve, i.e., we determine exactly one curve from the family of curves, as graphed in Figure 6.1.

Finding a particular antiderivative is also very useful in kinematics. For instance, if we know the velocity function $s'(t) = v(t)$, we can find the position function s if we are given one position datum to fix the constant. With the understanding that the constant is to be determined, it is often written:

$$s(t) = \int s'(t) dt = \int v(t) dt. \quad (6.17)$$

A common datum to prescribe is that $s(0) = s_0$ (where s_0 is some fixed number), but any data which “pins down” the function will suffice.

Example 6.1.5 Suppose $v = t^2 + 11t - 25$, and $s(1) = 4$. Find $s(t)$.

Solution:

$$s(t) = \int v(t) dt = \int (t^2 + 11t - 25) dt = \frac{t^3}{3} + \frac{11t^2}{2} - 25t + C.$$

Using $s(1) = 4$ we get

$$\frac{1^3}{3} + \frac{11 \cdot 1^2}{2} - 25(1) + C = 4 \iff \frac{1}{3} + \frac{11}{2} - 25 + C = 4,$$

and so $C = 4 + 25 - \frac{11}{2} - \frac{1}{3} = 29 - \frac{35}{6} = \frac{174}{6} - \frac{35}{6} = \frac{139}{6}$. Finally, this gives us

$$s(t) = \frac{t^3}{3} + \frac{11t^2}{2} - 25t + \frac{139}{6}.$$

Now we will derive a well-known formula of physics.

Example 6.1.6 Suppose that acceleration is given by a constant, say $s''(t) = a$ (where a is fixed). Suppose further that $s(0) = s_0$ and $v(0) = v_0$. Now we work “backwards” from the acceleration towards the position function as follows:

$$v(t) = s'(t) = \int s''(t) dt = \int a dt = at + C_1.$$

(Note that the last computation required that acceleration, a , be constant.) Using $v(0) = v_0$, we then have

$$a \cdot 0 + C_1 = v_0 \iff C_1 = v_0.$$

This gives us the following equation, which itself is well known to physics students:

$$v(t) = at + v_0. \tag{6.18}$$

Now we integrate (6.18):

$$s(t) = \int s'(t) dt = \underbrace{\int v(t) dt}_{(6.18)} = \int (at + v_0) dt = a \cdot \frac{t^2}{2} + v_0t + C_2.$$

Finally, using $s(0) = s_0$, we get

$$a \cdot \frac{0^2}{2} + v_0(0) + C_2 = s_0 \implies C_2 = s_0.$$

Thus

$$s = \frac{1}{2}at^2 + v_0t + s_0. \tag{6.19}$$

It is important to note that (6.19) followed under the special condition that acceleration is constant (such as occurs when an object is in freefall in a constant gravitational field, with no other resistance). Nonconstant acceleration will not give (6.18) or (6.19). However, the method for computing v and s , given a , is the same when a is not constant:

1. find $v(t) = \int a(t) dt$, using one datum regarding velocity at a particular time, to fix the constant of integration;
2. find $s(t) = \int v(t) dt$, using another datum regarding position at a particular time, to fix the second constant of integration.

Actually, two position data can fix the constants as well, since we can just carry the first constant into the second calculation, and then we will have two equations with two unknowns (the constants of integration), and then solve for both constants.

Example 6.1.7 Suppose $a(t) = 3t^2$, $s(0) = 3$ and $s(1) = 5$. Find $v(t)$ and $s(t)$.

Solution: First we will find $v(t)$, to the extent that we can:

$$v(t) = \int a(t) dt = \int 3t^2 dt = 3 \cdot \frac{t^3}{3} + C_1 = t^3 + C_1.$$

Next we find the form of $s(t)$:

$$s(t) = \int v(t) dt = \int (t^3 + C_1) dt = \frac{t^4}{4} + C_1 t + C_2.$$

So we know that $s(t) = \frac{1}{4}t^4 + C_1 t + C_2$, for some C_1, C_2 . Using the facts that $s(0) = 3$ and $s(1) = 5$, we get the following system of two equations in two unknowns:

$$\begin{cases} \frac{0^4}{4} + C_1(0) + C_2 = 3 \\ \frac{1^4}{4} + C_1(1) + C_2 = 5 \end{cases} \iff \begin{cases} C_2 = 3 \\ C_1 + C_2 = \frac{19}{4} \end{cases}$$

From the second form of the system, we see $C_2 = 3$, and so $C_1 = \frac{19}{4} - 3 = \frac{7}{4}$. Putting all this together we first get

$$s(t) = \frac{t^4}{4} + \frac{7t}{4} + 3,$$

from which we can calculate $v(t) = s'(t)$ (or just read $v(t)$ off of our first integral calculation, inserting $C_1 = \frac{7}{4}$) to get

$$v(t) = t^3 + \frac{7}{4}.$$

6.1.4 First Trigonometric Rules

With every derivative formula for functions comes an analogous antiderivative formula, which is more or less the derivative formula in reverse. Sometimes the reverse is more obvious than other times. For instance, the power rule formula for derivatives is sometimes seen algorithmically as “*multiply* by the exponent (‘bring the power down’) and *decrease* the exponent by one,” as in

$$\frac{d}{dx} [x^n] = n \cdot x^{n-1}.$$

If we are careful to reverse the process, we need to do the inverse steps in reverse order: *increase* the exponent by one, and then *divide* by the exponent. That is the essence of (6.5):

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

By other sophisticated arguments, in the next section we will see a kind of reverse chain rule. A bit later in the text we will also come across what can be loosely called a reverse product rule called integration by parts, although there really is no good analog of the product rule with

integrals *per se*.^{7 8}

The formulas presented in this subsection are immediate consequences of our trigonometric derivative formulas. For instance, we have the following pair of formulas:

$$\begin{aligned}\frac{d \sin x}{dx} = \cos x &\iff \int \cos x \, dx = \sin x + C, \\ \frac{d \cos x}{dx} = -\sin x &\iff \int (-\sin x) \, dx = \cos x + C.\end{aligned}$$

This second integration formula is more awkward than necessary, since it is more likely we would like an antiderivative for $\sin x$ directly. We could multiply both sides by -1 , and rename the new constant C , or just notice that $\frac{d}{dx}(-\cos x) = \sin x$, to come to the formula

$$\int \sin x \, dx = -\cos x + C.$$

As before, we can always check these by taking the derivative of the right-hand side. Recalling our six basic trigonometric derivative formulas, and making adjustments for negative sign placements as above, we have the following pairs of derivative/integral formulas:

$$\frac{d \sin x}{dx} = \cos x \iff \int \cos x \, dx = \sin x + C, \quad (6.20)$$

$$\frac{d \cos x}{dx} = -\sin x \iff \int \sin x \, dx = -\cos x + C, \quad (6.21)$$

$$\frac{d \tan x}{dx} = \sec^2 x \iff \int \sec^2 x \, dx = \tan x + C, \quad (6.22)$$

$$\frac{d \cot x}{dx} = -\csc^2 x \iff \int \csc^2 x \, dx = -\cot x + C, \quad (6.23)$$

$$\frac{d \sec x}{dx} = \sec x \tan x \iff \int \sec x \tan x \, dx = \sec x + C, \quad (6.24)$$

$$\frac{d \csc x}{dx} = -\csc x \cot x \iff \int \csc x \cot x \, dx = -\csc x + C. \quad (6.25)$$

With these and our previous rules, we have some limited ability to compute integrals involving trigonometric functions.

Example 6.1.8 Consider the following integrals:

⁷Integration by parts is really an integration technique which takes advantage of the product rule for derivatives—or more precisely a permutation of the product rule for derivatives—but is not itself a product rule for integrals; it does not by itself give a formula for $\int f(x)g(x) \, dx$. Instead it gives a formula which can be summarized by

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int g(x)f'(x) \, dx,$$

which follows from integrating—i.e., applying $\int(\dots) \, dx$ to both sides of—the following rearrangement of the product rule,

$$f(x)g'(x) = [f(x)g(x)]' - g(x)f'(x).$$

Because the product rule for derivatives is what makes the technique of integration by parts valid, many authors describe it as a kind of analog of the product rule, though again, it is not a direct formula for the integral of a product like we had for the derivative of a product.

Still, it is a very useful technique which we will spend some time developing in a later chapter, when we have other methods to draw upon for the inevitable intermediate computations.

⁸There will be still several other techniques which are not at all simple reverses of derivative rules, and for which checking by differentiating (computing the derivative of) the answer is as difficult as, or more difficult than, the integration technique itself. Those sophisticated techniques are for later chapters.

- $\int \left[x^2 + \sin x - \frac{1}{x} \right] dx = \frac{x^3}{3} - \cos x + \ln |x| + C$
- $\int \cos w \, dw = \sin w + C$
- $\int \frac{\sin x}{\cos^2 x} dx = \int \left[\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \right] dx = \int \sec x \tan x \, dx = \sec x + C$
- $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C.$

In fact we are fortunate if a trigonometric integral has a form which is just the derivative of one of the six basic trigonometric functions. When it is the case, it often requires some rewriting, as in the latter pair of integrals above.

6.1.5 Integrals Yielding Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C, \quad (6.26)$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C, \quad (6.27)$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} |x| + C. \quad (6.28)$$

Note how we only employ three of the six arc-trigonometric functions in (6.26)–(6.28). In fact these are sufficient. Recall for instance that the arccosine and arcsine have derivatives which differ by the factor -1 . For simplicity, it is much more commonly written $\int \frac{-1}{\sqrt{1-x^2}} dx = -\sin^{-1} x + C$, rather than using the arccosine function as the antiderivative, i.e., $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$ though the latter is certainly legitimate. Indeed, since $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, we see $\cos^{-1} x$ and $-\sin^{-1} x$ differ by a constant. In fact one could rewrite (6.26) as $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + C$. The choice can sometimes depend upon which range of angles we wish the antiderivative function to output, though for this case we can adjust that with the constant C .

Similarly one usually writes $\int \frac{-1}{x^2+1} dx = -\tan^{-1} x + C$, though $\cot^{-1} x + C$ (for a “different” C) is also legitimate. Analogously for arcsecant and arccosecant; we usually avoid the arccosecant function as an antiderivative.

But for these last two there is another small complication. Note how Equation (6.28) has the absolute value on the antiderivative rather than the x -term of the denominator in the integrand, so it does not appear to be just a restatement of the derivative rule for the arcsecant: $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$. To see that (6.28) is still correct, note that $|x| = x$ if $x > 0$, and $|x| = -x$ if $x < 0$. Taking the derivative of $\sec^{-1} |x|$ for those two cases, as we did in the computation of $\frac{d}{dx} \ln |x|$ (see page 359) we can see that we do get $\frac{1}{x\sqrt{x^2-1}}$ both times. But it should also be noted that, while not often seen, it would be legitimate to have the absolute value inside, rather than outside, the integral, as in $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$.⁹

⁹To further complicate things, we could notice that $\sec^{-1} x = \cos^{-1} \frac{1}{x}$, so it can occur that computational software will output an arccosine function, as in $\int \frac{1}{x\sqrt{x^2-1}} dx = \cos^{-1} \frac{1}{|x|} + C$, and then this can be rewritten (with a “different C ”) $\int \frac{1}{x\sqrt{x^2-1}} dx = -\sin^{-1} \frac{1}{|x|} + C$. The software might also omit the absolute values, theoretically assuming $x > 0$. Still, the standard written computation would output the expected $\sec^{-1} |x| + C$.

Before listing some examples, we last make note of the convention mentioned earlier (see page 403), of using dx as a factor. So our new integration formulas are often written:

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x + C, \\ \int \frac{dx}{x^2+1} &= \tan^{-1} x + C, \\ \int \frac{dx}{x\sqrt{x^2-1}} &= \sec^{-1} |x| + C.\end{aligned}$$

The above formulas will become much more important in future sections. For now it is important to realize that these particular function forms do have (relatively) simple antiderivatives. At this point in the development we are not prepared to make full use of these forms, but we need to be aware of them. Sometimes a simple manipulation produces a function containing one of these forms.

Example 6.1.9 Compute $\int \frac{x^2}{x^2+1} dx$.

Solution: With the aid of long division, we can see that¹⁰

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1},$$

and so

$$\int \frac{x^2}{x^2+1} dx = \int \left[1 - \frac{1}{x^2+1} \right] dx = x - \tan^{-1} x + C.$$

6.1.6 Integrals Yielding Exponential Functions

To finish our list of integrals which arise from differentiation formulas, we list those yielding exponential functions. Below $a \in (0, 1) \cup (1, \infty)$.

$$\int e^x dx = e^x + C, \quad (6.29)$$

$$\int a^x dx = \frac{a^x}{\ln a} + C. \quad (6.30)$$

The first of these, (6.29) is the more obvious. To see (6.30), recall that $\frac{da^x}{dx} = a^x \ln a$, and that $\ln a$ is a constant. Thus

$$\frac{d}{dx} \left[\frac{a^x}{\ln a} \right] = \frac{1}{\ln a} \cdot \frac{da^x}{dx} = \frac{1}{\ln a} \cdot a^x \ln a = a^x, \quad \text{q.e.d.}$$

¹⁰A popular alternative technique for a fraction like that in our integrand is to strategically add and subtract a term in the numerator, which produces a term in the numerator identical to (or a multiple of) the denominator, and the extra term, from which we can make two fractions:

$$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} = 1 - \frac{1}{x^2+1}.$$

While this is a very useful technique for such a simple case, it does not easily extend to more complicated cases. Long division—when the degree of the numerator is at least that of the denominator—is more straightforward. However, we will eventually have need of a technique similar to that given above in this footnote, though that setting will be much more complicated and we will need all other advanced integration techniques to make full use of it.

Example 6.1.10 We compute some antiderivatives involving these.

- $\int [1 + x + e^x] dx = x + \frac{x^2}{2} + e^x + C,$
- $\int 2^x dx = \frac{2^x}{\ln 2} + C,$
- $\int \frac{2^x - 3^x}{5^x} dx = \int \left[\frac{2^x}{5^x} - \frac{3^x}{5^x} \right] dx = \int \left[\left(\frac{2}{5} \right)^x - \left(\frac{3}{5} \right)^x \right] dx = \frac{\left(\frac{2}{5} \right)^x}{\ln \frac{2}{5}} - \frac{\left(\frac{3}{5} \right)^x}{\ln \frac{3}{5}} + C,$
- $\int (1 + 3^x)^2 dx = \int [1 + 2 \cdot 3^x + 3^{2x}] dx = \int [1 + 2 \cdot 3^x + 9^x] dx = x + \frac{2 \cdot 3^x}{\ln 3} + \frac{9^x}{\ln 9} + C.$

In the last integral, we used the fact that $(3^x)^2 = 3^{x \cdot 2} = 3^{2 \cdot x} = (3^2)^x = 9^x$.

Exercises

1. Verify the following antiderivative formulas by differentiation (applying $\frac{d}{dx}$ to the right-hand sides).

$$(a) \int e^{2t} dt = \frac{1}{2}e^{2t} + C$$

$$(b) \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + C$$

$$(c) \int \cos 2x dx = \frac{1}{2} \sin 2x + C$$

$$(d) \int \ln x dx = x \ln x - x + C$$

$$(e) \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$(f) \int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

$$(g) \int \sin^2 x dx = \frac{1}{2}(x - \sin x \cos x) + C$$

$$(h) \int \sec 7x \tan 7x dx = \frac{1}{7} \sec 7x + C$$

$$(i) \int \sec x dx = \ln |\sec x| + C$$

2. Calculate the following integrals, in most cases by rewriting the integrand to achieve a form found earlier in the section.

$$(a) \int \frac{1}{x^{2/3}} dx$$

$$(e) \int \sqrt{9w} dw$$

$$(b) \int \frac{1}{\cos^2 x} dx$$

$$(f) \int (-5 \sin x) dx$$

$$(c) \int (x^2 + 3x + 9) dx$$

$$(g) \int 10^x dx$$

$$(d) \int (x^2 + 1)^3 dx$$

$$(h) \int \cot^2 t dt$$

3. Find two possible antiderivatives for $-1/(x^2 + 1)$, i.e., find two possible formulas for

$$\int \frac{-1}{x^2 + 1} dx.$$

4. For each, find the function with the given properties.

$$(a) f'(x) = 3x^2 + 2x + 5, \quad f(1) = 2.$$

$$(c) s'(t) = \frac{1}{t^2 + 1}, \quad s(1) = \pi/2.$$

$$(b) g'(z) = \sqrt{z}, \quad g(4) = \frac{19}{3}.$$

$$(d) f'(x) = 1 + \cos x, \quad f(\pi/2) = 6.$$

6.2 Sigma-Notation

This short section looks at the Σ -notation common for signifying summations. For example,

$$\sum_{i=1}^n a_i = \underbrace{a_1}_{i=1} + \underbrace{a_2}_{i=2} + \underbrace{a_3}_{i=3} + \cdots + \underbrace{a_{n-1}}_{i=n-1} + \underbrace{a_n}_{i=n}. \quad (6.31)$$

It is no accident that the “Greek S” is used to signify a sum. In the above, i is the *index* of summation, ranging along the integers from 1 to n , i.e., $i = 1, 2, 3, \dots, n$. If $a_1 = 3$, $a_2 = 5$, $a_3 = 1$, $a_4 = 7$, and $a_5 = -2$, we can write

$$\begin{aligned} \sum_{i=1}^5 a_i &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 3 + 5 + 1 + 7 - 2 = 14. \end{aligned}$$

Often the summation desired is of the form $\sum_{i=1}^n f(i)$, where $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$, though the domain of f need not begin at $i = 1$. For instance,

$$\begin{aligned} \sum_{i=1}^5 i^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 = 55, \\ \sum_{i=0}^3 (2i+1) &= (2(0)+1) + (2(1)+1) + (2(2)+1) + (2(3)+1) \\ &= 1 + 3 + 5 + 7 = 16, \\ \sum_{i=0}^4 2 &= 2 + 2 + 2 + 2 + 2 \\ &= 5(2) = 10. \end{aligned}$$

Some care must be taken regarding the range of values for the *index*, which in the above is i for each example. Other indices are also common, usually lower-case Latin or Greek symbols.

The notation has many uses. It can be found nearly anywhere a large number of similar numbers are routinely added, such as in accounting or on spreadsheets. In statistics, if we have data x_1, x_2, \dots, x_n , then the *sample mean* is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n} \sum_{i=1}^n x_i,$$

while the sample deviation is then given by

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}.$$

Some arithmetic properties of summations can be seen with little difficulty. In particular, we

have

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i, \quad (6.32)$$

$$\sum_{i=1}^n (ka_i) = k \cdot \sum_{i=1}^n a_i \quad (6.33)$$

$$\sum_{i=1}^n k = nk. \quad (6.34)$$

The first is simply a regrouping, the second the distributive law, while the third is a simple counting of terms added. Note that $\sum_{i=0}^n k = (n+1)k$, while $\sum_{i=2}^5 k = 4k$, so again some care must be taken in counting the number of terms in the summation.

It is interesting to note that there are formulas which one can derive for sums of positive powers of the index. The first few are as follow:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}. \quad (6.35)$$

We will leave the computations of these to the exercises. With these, we can compute for example

$$\sum_{i=1}^5 i^2 = \frac{5(5+1)(2(5)+1)}{6} = \frac{5 \cdot 6 \cdot 11}{6} = 55,$$

as we found in our computation on the previous page. With these latest formulas, together with (6.32)–(6.34), we can more readily compute some summations without resorting to adding each individual term.

Example 6.2.1 Consider the following computations.

$$\begin{aligned} \sum_{i=1}^{10} (3 + i^2) &= \sum_{i=1}^{10} 3 + \sum_{i=1}^{10} i^2 = 10 \cdot 3 + \frac{10(10+1)(2(10)+1)}{6} \\ &= 30 + \frac{10 \cdot 11 \cdot 21}{6} = 30 + 385 = 415, \end{aligned}$$

$$\sum_{i=1}^{100} i = \frac{100(100+1)}{2} = 5050,$$

$$\sum_{i=50}^{100} i^2 = \sum_{i=1}^{100} i^2 - \sum_{i=1}^{49} i^2 = \frac{(100)(101)(201)}{6} - \frac{(49)(50)(99)}{6} = 338,350 - 40,425 = 29,7925.$$

The last summation we were able to rewrite as a difference, the terms we do not need having cancelled

6.3 Fundamental Theorem of Calculus: First Look

Anytime a theorem is called “fundamental” in its field, we expect it to be somewhat deep, ultimately intuitive, very important, and not trivial to prove. This is the case with the fundamental theorem of calculus (FTC), as discussed here. An actual proof of the theorem is beyond the scope of this text.¹¹

Instead we present a case where it is, more or less, obvious, and then generalize somewhat. Along the way there are several concepts to define and explore, and the explanation bears careful study and several revisits.

We begin with the twin concepts of relative and percent error, show how they propagate in summations, show how the study of approximate displacements leads us to Riemann Sums and one case of the FTC, and then generalize for the full conclusion of one part of the FTC. There is another part, which we leave for a later section.

6.3.1 Absolute, Relative and Percent Errors

For a simple example of these three types of errors, consider a man who weighs 200 lbs, weighed on a scale which indicates his weight to be 210 lbs. In such a case we say the *absolute error* is 10 lbs. For a 200 lbs man this seems “relatively” small, but if we are weighing a newborn child, that is clearly an unacceptable error. Thus it is also important to note what fraction of his weight the error represents, so we compute the *relative error*, namely $(10 \text{ lbs})/(200 \text{ lbs})$, or 0.05. Now as a percentage, we multiply by 100% (which is just another expression for 1) and find the percent error to be 5%. Note that the relative and percent error are unitless, in the sense that the “lbs” cancel.

Put colloquially, these errors are defined as follow:

$$\text{Absolute Error} = (\text{Actual Quantity}) - (\text{Measured Quantity}) \quad (6.36)$$

$$\text{Relative Error} = (\text{Absolute Error})/(\text{Actual Quantity}) \quad (6.37)$$

$$\text{Percent Error} = (\text{Relative Error}) \cdot 100\%. \quad (6.38)$$

Some texts will define the absolute error to include absolute values of the quantity on the right-hand side of (6.36), hence the name. One can then also define relative absolute error, and percent absolute error, in the obvious ways. However in all cases it is informative to have a sign (+/−) associated to the error, but we will still use the phrase *error* rather than absolute (in the sense of absolute values) error, when there is no confusion.

Relative and percent errors are easily visualized, and in fact judging when a relative or percentage error is “small” or “large” is fairly easy given an accurate illustration of the quantities involved.

Relative and percent errors are easily visualized, and in fact judging when a relative or percentage error is “small” or “large” is fairly easy given an accurate illustration of the quantities involved. See Figure 6.2.

Now we consider what would be the cumulative effect on a summation if the measured amount were consistently a given percentage higher than the actual.

¹¹All 3-semester calculus textbook authors attempt an argument for why the FTC is true. Some give partial proofs which are as intuitive as possible, while others give proofs that are more technical, but closer to an actual proof. So far, none have offered a complete proof without having one large gap which requires junior or senior level Real Analysis to fill. This textbook is no different. Here we opt for an intuitive arguments, and later outline a proof which is closer to an actual proof, but neither the intuitive, nor the more technical, of the arguments given here constitute a rigorous proof. That is left for junior or senior level classes.

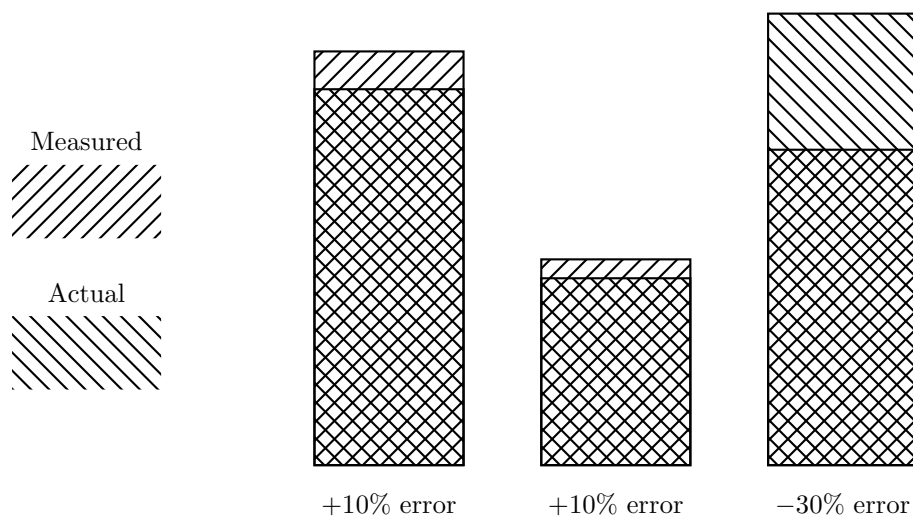


Figure 6.2: Visual illustration of three examples of percent errors, the first two being equal as percentages, the latter being three times as large as a percent error.

Example 6.3.1 Suppose the actual quantity we desire to know is $\sum_{i=1}^n a_i$, where we attempt to measure each of the a_i , and each is measured to be b_i , where b_i is a 5% overestimation of a_i in each case. Then the measured summation will be

$$\sum_{i=1}^n b_i = \sum_{i=1}^n (1.05a_i) = 1.05 \sum_{i=1}^n a_i.$$

In other words, if the b_i all overestimate the respective a_i by 5%, then the summation of the b_i overestimates the summation of the a_i by exactly 5%.

The example above simply illustrates the distributive property of multiplication over summations. We can conclude similarly that a consistent underestimation of a_i by 5% would result in exactly a 5% underestimation of the summation. More generally, since these would be the “extreme” cases, we can further state that if, in the sense of absolute values, the percent absolute error in a_i is less than 5% (+/-), then the percent absolute error in the sum must also be less than 5% (+/-). This leads us to the more general conclusion:

Theorem 6.3.1 If each a_i is estimated by a respective b_i within $p\%$ error, then we have $\sum a_i$ is estimated within $p\%$ error by $\sum b_i$.

That fact will be important in the next step of our argument for the validity of the FTC.

6.3.2 A Physics Example

Here we consider an abstract motion problem. We wish to find the net displacement of an object in one-dimensional motion, over the time interval $[t_0, t_f]$. For a classical problem, the velocity $v(t)$ over this time interval should be continuous. For technical reasons explained later, we also assume that it is positive, i.e., $v(t) > 0$ on $[t_0, t_f]$.

Now the actual net displacement over the time interval is given by $s(t_f) - s(t_0)$. Intuitively it will also be positive.

Now we consider a scheme for approximating this net displacement, based upon the velocity function. We do this by partitioning the time interval into subintervals with endpoints

$$t_0 < t_1 < t_2 < t_3 < \cdots < t_{n-1} < t_n = t_f.$$

The width of the i th subinterval $[t_{i-1}, t_i]$ will then be

$$\Delta t_i = t_i - t_{i-1}. \quad (6.39)$$

If that interval is short enough, then the velocity change over the interval will be small and the net displacement over that interval can be approximated by

$$s(t_i) - s(t_{i-1}) \approx v(t_i^*)\Delta t_i,$$

where $t_i^* \in [t_{i-1}, t_i]$ is a point in the interval at which we *sample* the velocity.

Our scheme is thus to approximate the net displacement on each interval, and sum these.

Interval		Sample	Width	Approximate Displacement		Actual Displacement
$[t_0, t_1]$	\ni	t_1^*	Δt_1	$v(t_1^*)\Delta t_1$	\approx	$s(t_1) - s(t_0)$
$[t_1, t_2]$	\ni	t_2^*	Δt_2	$v(t_2^*)\Delta t_2$	\approx	$s(t_2) - s(t_1)$
$[t_3, t_2]$	\ni	t_3^*	Δt_3	$v(t_3^*)\Delta t_3$	\approx	$s(t_3) - s(t_2)$
\vdots		\vdots	\vdots	\vdots	\vdots	\vdots
$[t_{n-2}, t_{n-1}]$	\ni	t_{n-1}^*	Δt_{n-1}	$v(t_{n-1}^*)\Delta t_{n-1}$	\approx	$s(t_{n-1}) - s(t_{n-2})$
$[t_{n-1}, t_n]$	\ni	t_n^*	Δt_n	$v(t_n^*)\Delta t_n$	\approx	$s(t_n) - s(t_{n-1})$

When we now sum the last two columns, respectively, we get much cancellation in the last column, resulting in the approximation

$$\sum_{i=1}^n v(t_i^*)\Delta t_i \approx s(t_f) - s(t_0). \quad (6.40)$$

However it is not clear how good the above approximation actually is. For that we turn to our earlier note, that it is reasonable we can choose intervals small enough that the velocity changes no more than p % (+/-), for any $p > 0$. In doing so, we are assured that the net displacement over each interval is within p percent of the actual for that interval, and so by our Theorem 6.3.1, page 417, the sum on the left of (6.40) is within p percent. Now we reason that the percent error will shrink to zero as the interval lengths shrink to zero, and we will argue that¹²

$$\lim_{\max \Delta x_i \rightarrow 0^+} \sum_{i=1}^n v(t_i^*)\Delta t_i = s(t_f) - s(t_0). \quad (6.41)$$

At this point we note that $s(t)$ is an antiderivative of $v(t)$, which is no accident.

¹²There is one caveat to this reasoning, which is that on an interval where the velocity may be momentarily zero, our choice of $v(t_i^*)$ could be off by 100%, and if the actual displacement were zero and we chose some t_i^* such that $v(t_i^*) \neq 0$, then our error is an infinite percent. Thus we have to rely on the fact that we can then choose the interval small enough that the absolute error is as small as we like, and keep the percentage error small in the other intervals. This is partially alleviated by our assumption that $v(t) > 0$ on $[t_0, t_f]$, but we would like our analysis to work under less restrictive conditions. Later illustrations will help show that this ability is reasonable.

6.3.3 General Riemann Sums, FTC

The sum on the right-hand side of (6.40) is one example of what is known as a *Riemann Sum*. More generally, for $f(x)$ defined on $[a, b]$, we define Riemann sums to be any sum of the form

$$\sum_{i=1}^n f(x_i^*) \Delta x_i, \quad (6.42)$$

where we *partition* $[a, b]$ into subintervals with endpoints

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b,$$

and $\Delta x_i = x_i - x_{i-1}$ is the width of the i th subinterval. What we have argued in the context of velocities is actually one part of the Fundamental Theorem of Calculus:

Theorem 6.3.2 (Fundamental Theorem of Calculus, Part 1): *For $f(x)$ continuous on $[a, b]$, and $F(x)$ being an antiderivative of $f(x)$ on $[a, b]$, and Riemann sums as above, we have*

$$\lim_{\max\{\Delta x_i\} \rightarrow 0} \left(\sum_{i=1}^n f(x_i^*) \Delta x_i \right) = F(b) - F(a). \quad (6.43)$$

Note that $\max\{\Delta x_i\} \rightarrow 0 \implies n \rightarrow \infty$, so we shrink all the subintervals and therefore increase their number. At this point we introduce some notation.

$$\int_a^b f(x) dx \stackrel{\text{definition}}{=} \lim_{\max\{\Delta x_i\} \rightarrow 0} \left(\sum_{i=1}^n f(x_i^*) \Delta x_i \right), \quad (6.44)$$

$$F(x) \Big|_a^b \stackrel{\text{definition}}{=} F(b) - F(a). \quad (6.45)$$

With definitions (6.44) and (6.45), we can rewrite the Fundamental Theorem of Calculus as

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a). \quad (6.46)$$

To distinguish the integral symbol \int in this context from its use in Section 6.1, the quantity on the left-hand sides of (6.44) and (6.46) is called the *definite integral of $f(x)$ with respect to x , from $x = a$ to $x = b$* . Of course the FTC gives a very strong connection between the two uses of the symbol.¹³

Note that (6.46) requires that f be continuous on $[a, b]$, and F be an antiderivative there, in the sense of Definition 6.1.1, page 400.

The geometric interpretation of this limit

¹³The endpoints a and b in the definite integral are often referred to as the lower and upper *limits of integration*, perhaps an unfortunate term since “limit” usually refers to very different concepts. Perhaps better words in this context would be *boundary points* or similar.

6.4 Substitution With Power Rule

Substitution in general is the most important of the integrating techniques, finding its way into the other techniques as well. While we introduce it here, for now we limit the scope to power rules.

Before looking at this method formally, consider the following antiderivative statements, each of which refer to the same power rule (perhaps most familiar in the first case):

$$\begin{aligned}\int x^2 dx &= \frac{x^3}{3} + C, \\ \int u^2 du &= \frac{u^3}{3} + C, \\ \int (\sin x)^2 d(\sin x) &= \frac{(\sin x)^3}{3} + C.\end{aligned}\tag{6.47}$$

The last integral is simply asking for an antiderivative of $(\sin x)^2$ with respect to $\sin x$. Indeed, we can check the answer as before:

$$\frac{d}{d \sin x} \left[\frac{1}{3} (\sin x)^3 \right] = \frac{1}{3} \cdot 3 (\sin x)^2 = (\sin x)^2,$$

as we expect. Of course we usually take derivatives and antiderivatives with respect to a variable, and not a function. However the integral in (6.47) is not so unlikely to occur as one might think. Recall that $df(x) = f'(x) dx$ is the definition of the differential (see (5.13), page 394). Thus $d \sin x = \cos x dx$, and the integral in (6.47) can be written instead

$$\int (\sin x)^2 \cos x dx = \int (\sin x)^2 \frac{d \sin x}{dx} dx = \int (\sin x)^2 d \sin x = \frac{1}{3} (\sin x)^3 + C.$$

Indeed, it is not hard to see that the chain rule gives us $\frac{d}{dx} \left[\frac{1}{3} (\sin x)^3 \right] = \frac{1}{3} \cdot 3 (\sin x)^2 \cos x = \sin^2 x \cos x$.

In this section we will concentrate on integrals of the form

$$\int u^n du = \begin{cases} \frac{1}{n+1} \cdot u^{n+1} + C & \text{if } n \neq -1, \\ \ln |u| + C & \text{if } n = -1. \end{cases}\tag{6.48}$$

As anticipated in the discussion above, the content of the differential du may be more expansive than what we may expect from a single variable. The point of this section is to recognize when we have the form (6.48), and how to go about rewriting the integral into the proper form.

The reader should be forewarned: this method requires a fair amount of practice. It is not a simple algorithm. For each problem, the reader has to decide which substitution will produce an integral which can be computed with known rules. Here we will limit ourselves to the power rule (6.48), but in subsequent sections we will delve into many other rules, and it is not always obvious which rule should be used for a given integral. With practice one learns to look for clues, and anticipate what will occur several steps ahead, to see if there is indeed an integration rule which can apply.¹⁴

¹⁴In fact, often there is no rule which will produce an antiderivative, and then some approximation scheme will be necessary. Still, it is most desirable to have an exact antiderivative, and we can find one often enough that it is well worth studying these techniques.

6.4.1 The Technique

Here we will look at some of the simpler problems of integration by substitution. As we proceed, several observations will be made regarding the method.

Example 6.4.1 Compute the indefinite integral $\int (x^2 + 1)^7 \cdot 2x \, dx$.

Solution: The technique is to introduce a new variable, u , with which we can write the original integral in a simpler form. We also have to take into account what will be the new differential, namely du :

$$\begin{aligned} u &= x^2 + 1 \\ \implies du &= 2x \, dx. \end{aligned}$$

(Recall that if u is a function of x , then $du = u'(x) \, dx$, consistent with $\frac{du}{dx} = u'(x)$.) Using this information, we can replace all the terms in the original integral: the $(x^2 + 1)^7$ becomes u^7 , and the terms $2x \, dx$ collectively become du (see the above implication arrow). Thus

$$\int (x^2 + 1)^7 \cdot \underline{2x \, dx} = \int u^7 \underline{du} = \frac{1}{8} u^8 + C.$$

This is all true, but **we** introduced u , while the original question asked for an antiderivative with respect to x . We only need to replace u in the final answer, using again $u = x^2 + 1$. Summarizing,

$$\int (x^2 + 1)^7 \cdot 2x \, dx = \int u^7 \, du = \frac{1}{8} u^8 + C = \frac{1}{8} (x^2 + 1)^8 + C.$$

Note that we can check our answer in the above example by computing the derivative of the answer (using the chain rule), yielding $\frac{d}{dx} \left[\frac{1}{8} (x^2 + 1)^8 \right] = \frac{1}{8} \cdot 8(x^2 + 1)^7 \cdot 2x = (x^2 + 1)^7 \cdot 2x$ as hoped. In fact, integration by substitution, at least in its simplest forms, is often called a type of reverse chain rule. Indeed, we can rewrite (6.48) as follows:

$$\int u^n \, du = \int u^n \cdot \left(\frac{du}{dx} \right) dx = \int \left[u^n \cdot \frac{du}{dx} \right] dx = \begin{cases} \frac{1}{n+1} \cdot u^{n+1} + C & \text{if } n \neq -1, \\ \ln |u| + C & \text{if } n = -1. \end{cases} \quad (6.49)$$

To see that this is correct, if we take the derivative of the answers *with respect to* x , we see that we do indeed get $u^n \cdot \frac{du}{dx}$ from the chain rule.

In short, with integration by substitution we try to pick some function we call u , so that

1. a main part of the integrand can be written as a simple function of u —one for which we know the antiderivative with respect to u —and, equally crucial, so that
2. the remaining variable terms of the integral can be safely absorbed in du (except for multiplicative constants, which we will see add only a slight complication).

If these are both satisfied, our substitution of u and du terms gives us a new, simple integral (entirely in terms of u and du !).

When working such a problem (as opposed to, say, *publishing* a problem and solution for professional consumption), a useful format is to (1) write the original integral, (2) write the substitution function u , with its differential du both on different lines than the original integral, (3) write the new form of the original integral, i.e., in u and du , (4) compute the antiderivative

of this new integral in u as a continuation of the first step, and (5) resubstitute to arrive at the antiderivative in x . Hence a typical homework-style presentation of Example 6.4.1 might look like the following (with the choice of u and resulting du offset and below the original integral):

$$\int (x^2 + 1)^7 \cdot \underline{2x \, dx} = \int u^7 \, \underline{du} = \frac{1}{8}u^8 + C = \frac{1}{8}(x^2 + 1)^8 + C$$

$$\begin{array}{l} u = x^2 + 1 \\ \implies du = \underline{2x \, dx} \end{array} \left| \right.$$

That part of the integral which we hope to absorb into du is underlined in the original integral, the computation of du , and the corresponding term in the new integral. The rest of the integral was just $(x^2 + 1)^7 = u^7$. In fact, when we choose u so that a major portion of the integral can be written u^n , then any other factors which are variable, along with the differential dx , must be absorbed in du or the substitution will fail (because the resulting integral will contain both x and u and no antiderivative rules will apply). We will continue to use this kind of spatial organization when we integrate by substitution in the examples below.

Example 6.4.2 Compute the indefinite integral $\int \sin^2 x \cos x \, dx$.

Solution: Note that this integral can be written $\int (\sin x)^2 \cos x \, dx$. Now we proceed:

$$\int \sin^2 x \, \underline{\cos x \, dx} = \int u^2 \, \underline{du} = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$$

$$\begin{array}{l} u = \sin x \\ \implies du = \underline{\cos x \, dx} \end{array} \left| \right.$$

Before continuing we will make a very minor change to the integral in the first example (Example 6.4.1, page 421), and show a simple way to extend our method to handle this.

Example 6.4.3 Compute $\int x(x^2 + 1)^7 \, dx$.

Solution: Here we will make the same substitution as before, but the du will have an extra factor of 2. Since constant factors are relatively easy to handle in derivative and antiderivative problems in general, we should not expect this extra factor of 2 to cause much difficulty. It will simply mean one extra step in the substitution computations.¹⁵

¹⁵There is another method used by some texts to handle a problem such as this, which is to simply introduce the needed factor of 2 in the integral to complete the differential $du = 2x \, dx$, and compensate for the insertion of the new factor by simultaneously inserting a factor of $\frac{1}{2}$, which is simply carried through the rest of the calculation:

$$\int x(x^2 + 1)^7 \, dx = \int \frac{1}{2} \underbrace{(x^2 + 1)^7}_{u^7} \cdot \underbrace{2x \, dx}_{du} = \frac{1}{2} \int u^7 \, du = \frac{1}{2} \cdot \frac{1}{8}u^8 + C = \frac{1}{16}(x^2 + 1)^8 + C,$$

where again $u = x^2 + 1$, $du = 2x \, dx$.

This method is appealing because one rewrites the integrand into a form where it is, more or less, clearly a derivative of a chain rule function (perhaps multiplied by a constant, as with $\frac{1}{2}$ here).

We will avoid this method because, though it is not so challenging for simpler problems, it quickly becomes unreasonably difficult if an integral is complicated. Furthermore, the method presented in this text—in the author's opinion—makes for much better preparation for more advanced methods, such as trigonometric substitution and integration by parts.

$$\int x(x^2 + 1)^7 dx = \int u^7 \cdot \frac{1}{2} du = \frac{1}{2} \cdot \frac{1}{8} u^8 + C = \frac{1}{16} (x^2 + 1)^8 + C$$

$$\begin{array}{l} u = x^2 + 1 \\ \implies du = 2x dx \\ \implies \frac{1}{2} du = x dx \end{array} \left| \right.$$

This time the extra nonconstant and differential terms of the original integral were, collectively, $x dx$. Though that product is not exactly du , it is a constant times du . In our substitution we took an extra step and solved, again collectively, for $x dx = \frac{1}{2} du$.

The preceding example shows that we need to be flexible when looking for a possible power rule application. Not every integral where we can use the power rule will be of the strict form (6.49), page 421. Indeed, we need to be especially vigilant to notice that an integral may be of the form

$$\int k \cdot u^n du = \int k \cdot u^n \cdot \left(\frac{du}{dx} \right) dx. \quad (6.50)$$

So when we make a substitution, we try not to be distracted by extra or missing multiplicative constants, as they will work themselves out in the substitution and final integration steps.

Example 6.4.4 Compute $\int x^3 \cos^5 x^4 \sin x^4 dx$.

Solution: It is perhaps more obvious how to proceed if we rewrite the integral in the form $\int x^3 (\cos x^4)^5 \sin x^4 dx$. Then we see that the u^n term will be $(\cos x^4)^5 = u^5$, where $u = \cos x^4$. Next we need to see if du can absorb the other nonconstant terms:

$$\int x^3 \cos^5 x^4 \sin x^4 dx = \int u^5 \left(-\frac{1}{4} \right) du = -\frac{1}{4} \cdot \frac{1}{6} u^6 + C = -\frac{1}{24} (\cos x^4)^6 + C.$$

$$\begin{array}{l} u = \cos x^4 \\ \implies du = -\sin x^4 \cdot 4x^3 dx \\ \implies -\frac{1}{4} du = x^3 \sin x^4 dx \end{array} \left| \right.$$

We should point out that the method above would not have worked without *both* the x^3 and the $\sin x^4$ terms in the integral, for the du would have variable terms not in the original integral.

Also, it is possible to compute the integral in the previous by using two substitution steps instead of one. For instance, a student recognizing that x^4 , and a multiple of its derivative in the form of x^3 , both appear, might first make a substitution of the form $u = x^4$:

$$\int x^3 \cos^5 x^4 \sin x^4 dx = \int \cos^5 u \sin u \frac{1}{4} du = \frac{1}{4} \int \cos^5 u \sin u du.$$

$$\begin{array}{l} u = x^4 \\ \implies du = 4x^3 dx \\ \implies \frac{1}{4} du = x^3 dx \end{array} \left| \right.$$

At this point, we have a simpler integral which itself requires a substitution:

$$\frac{1}{4} \int \cos^5 u \sin u \, du = \frac{1}{4} \int w^5 (-dw) = -\frac{1}{4} \cdot \frac{1}{6} w^6 + C = -\frac{1}{24} (\cos u)^6 + C.$$

$$\begin{array}{l} w = \cos u \\ \implies dw = -\sin u \, du \\ \implies -dw = \sin u \, du \end{array} \left| \right.$$

Of course this gives the answer in terms of u , so we substitute back again, in terms of x . Summarizing,

$$\int x^3 \cos^5 x^4 \sin x^4 \, dx = \dots = -\frac{1}{24} w^6 + C = -\frac{1}{24} \cos^6 u + C = -\frac{1}{24} \cos^6 x^4 + C.$$

The second approach is longer, but it has the advantage that we are not trying to rewrite the integral in one, all-encompassing (and thus more complicated) substitution step. Indeed it is sometimes desirable to simplify an integral with substitution even if the resulting integral cannot be evaluated immediately. With most examples we will use the first method, but the student working problems should be aware that the option of successive substitutions is perfectly valid.

Next we look at a few very common types of examples where the power of n is $1/2$, -1 and -2 . These appear often enough that it is worth some effort to remember them specifically.

Example 6.4.5 Compute $\int \frac{x}{\sqrt{x^2-9}} \, dx$.

Solution Here we will take $u = x^2 - 9$, since the $du = 2x \, dx$ can absorb both the dx and the extra factor of x :

$$\int \frac{x}{\sqrt{x^2-9}} \, dx = \int u^{-1/2} \cdot \frac{1}{2} \, du = \frac{1}{2} \cdot 2u^{1/2} + C = \sqrt{x^2-9} + C.$$

$$\begin{array}{l} u = x^2 - 9 \\ \implies du = 2x \, dx \\ \implies \frac{1}{2} du = x \, dx \end{array} \left| \right.$$

Example 6.4.6 Compute $\int \frac{\sin x}{\cos x} \, dx$.

Solution: Here we will take $u = \cos x$, since $du = -\sin x \, dx$ will absorb the other terms.

$$\int \frac{\sin x \, dx}{\cos x} = \int \frac{1}{u} (-du) = -\ln |u| + C = -\ln |\cos x| + C.$$

$$\begin{array}{l} u = \cos x \\ \implies du = -\sin x \, dx \\ \implies -du = \sin x \, dx \end{array} \left| \right.$$

Note that if we instead took $u = \sin x$, then $du = \cos x \, dx$, but $\cos x$ is not a **multiplicative** factor in the original integral; the desired factor is $\frac{1}{\cos x}$, which is not contained in the du term if $u = \sin x$.

It should be remembered that checking these antiderivatives is as simple as computing the derivative of the answer. Here

$$\frac{d}{dx} [-\ln |\cos x|] = -\frac{1}{\cos x} \cdot \frac{d}{dx} \cos x = -\frac{1}{\cos x} (-\sin x) = \frac{\sin x}{\cos x},$$

as we hope. Of course our original integrand, and the derivative above, can both be written $\tan x$.

Note that we can write $-\ln |\cos x| = \ln |(\cos x)^{-1}| = \ln |\sec x|$, so many calculus books contain the integration formula

$$\int \tan x \, dx = \ln |\sec x| + C. \quad (6.51)$$

(It is also interesting to “fill in the dots” for the computation $\frac{d}{dx} \ln |\sec x| = \dots = \tan x$, verifying (6.51). See also Exercises 15 and 16, page 430.)

Example 6.4.7 Compute $\int \frac{e^{3x}}{(e^{3x} + 4)^2} \, dx$.

Solution: Here we note that the numerator of the integrand, namely e^{3x} , is the derivative of $e^{3x} + 4$, except for a multiplicative constant. Thus we will let $u = e^{3x} + 4$:

$$\begin{aligned} \int \frac{e^{3x}}{(e^{3x} + 4)^2} \, dx &= \int \frac{1}{u^2} \cdot \frac{1}{3} \, du = \frac{1}{3} \int u^{-2} \, du = \frac{1}{3} \cdot (-1)u^{-1} + C \\ &\Rightarrow \left. \begin{array}{l} u = e^{3x} + 4 \\ du = \underbrace{e^{3x}} \cdot \underbrace{3 \, dx} \\ \frac{1}{3} du = \underbrace{e^{3x} \, dx} \end{array} \right| = -\frac{1}{3}(e^{3x} + 4)^{-1} + C = \frac{-1}{e^{3x} + 4} + C. \end{aligned}$$

At this point we notice three common forms of integration by substitution:

$$\int \frac{u'(x)}{\sqrt{u(x)}} \, dx = 2\sqrt{u(x)} + C, \quad (6.52)$$

$$\int \frac{u'(x)}{u(x)} \, dx = \ln |u(x)| + C, \quad (6.53)$$

$$\int \frac{u'(x)}{[u(x)]^2} \, dx = \frac{-1}{u(x)} + C. \quad (6.54)$$

In all three cases, $u'(x) \, dx = du$, and we have simple power rules. In the first and third cases there are multiplicative constants which occur. There is no real need to memorize these, but they occur often enough that their “mechanics” should become familiar. For that reason these three results can become, if not memorized, then at least easily cited.

The method also works for cases where the du term is just a constant multiple of dx :

Example 6.4.8 Compute $\int \frac{1}{(6 - 2x)^5} \, dx$.

Solution:

$$\begin{aligned} \int \frac{1}{(6 - 2x)^5} \, dx &= \int u^{-5} \cdot \frac{-1}{2} \, du = -\frac{1}{2} \cdot \frac{1}{-4} u^{-4} + C = \frac{1}{8}(6 - 2x)^{-4} + C \\ &\Rightarrow \left[\begin{array}{l} u = 6 - 2x \\ du = -2 \, dx \\ \frac{-1}{2} du = \, dx \end{array} \right] = \frac{1}{8(6 - 2x)^4} + C. \end{aligned}$$

In the case that $du = dx$, this can often be anticipated and the experienced calculus student might omit the middle steps:

Example 6.4.9 Compute $\int (x + 9)^4 dx$.

Solution:

$$\int (x + 9)^4 = \int u^4 du = \frac{1}{5}u^5 + C = \int \frac{1}{5}(x + 9)^5 + C.$$

$$\left. \begin{array}{l} u = x + 9 \\ \implies du = dx \end{array} \right\}$$

Another way to look at the example above is to realize that $d(x + 9) = dx$, so we can write

$$\int (x + 9)^4 dx = \int (x + 9)^4 d(x + 9) = \frac{1}{5}(x + 9)^5 + C.$$

In other words, dx is the same as $d(x + 9)$, so we get the same if we interpret the original integral as an antiderivative with respect to $(x + 9)$.¹⁶ Indeed this is a shortcut one learns with practice—thinking but perhaps not writing the second step—but at first it is still best to write out the full substitution, as in the example above, at least until one is proficient in the method as presented here. Of course this is the analog to a chain rule where the “inner” derivative is 1:

$$\frac{d}{dx} \left[\frac{1}{5}(x + 9)^5 \right] = \frac{1}{5} \cdot 5(x + 9)^4 \cdot \frac{d(x + 9)}{dx} = \frac{1}{5} \cdot 5(x + 9)^4 \cdot 1 = (x + 9)^4, \text{ q.e.d.}$$

6.4.2 A Slight Twist on the Method

Recall our second example, namely Example 6.4.3 on page 423: $\int x(x^2 + 1)^7 dx$. We used a substitution $u = x^2 + 1$ because $du = 2x dx$ contained the extra factor of x in the integrand. The substitution eventually gave us $\int u^7 \cdot \frac{1}{2} du$, which was a simple power rule. Of course we could have “simply” expanded the original function

$$\begin{aligned} x(x^2 + 1)^7 &= x(x^2 + 7x^4 + 21x^6 + 35x^8 + 35x^{10} + 21x^{12} + 7x^{14} + 1) \\ &= x^3 + 7x^5 + 21x^7 + 35x^9 + 35x^{11} + 21x^{13} + 7x^{15} + x, \end{aligned}$$

and integrated “term by term.” However the substitution method was arguably easier, and the answer’s simple form, $\frac{1}{16}(x^2 + 1)^8 + C$ would probably not be recognizable from a strategy which expands the integrand first.

Now consider the integral $\int x(x - 1)^{3/2} dx$. Here we can not simply “expand” the integrand (even by brute force, as above), because of the fractional power term $(x - 1)^{3/2}$, which is algebraically more difficult to deal with than positive integer powers. Furthermore, if we let $u = x - 1$, then $du = dx$, but this differential term cannot absorb the extra factor x . The key is to then notice that the original substitution offers a way out: that extra factor x can be rewritten $u + 1$ (since $u = x - 1 \iff u + 1 = x$). Below we show how this can be utilized. Indeed we will expand the new integrand, but what is interesting is how the algebraic difficulties of the $(x - 1)^{3/2}$ term (namely that this is of the form $(a + b)^r$, $r \notin \mathbb{N}$) is transferred to the x term which, being a positive integer power, is then easier to handle. Below we write this out in the standard example format:

¹⁶Note that the *change* in $x + 9$ is the same as the change in x .

Example 6.4.10 Compute $\int x(x-1)^{3/2} dx$.

Solution: Here we substitute for dx **and** x . Both substitutions are calculated below, but separately. (This time we underline the substitution for x , instead of the differential part.) Once the substitutions are completed, we can perform the multiplication to get two simple power rules:

$$\int \underline{x}(x-1)^{3/2} dx = \int \underbrace{(u+1)}_{u=x-1} u^{3/2} du = \int (u^{5/2} + u^{3/2}) du = \frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2} + C$$

$$\left. \begin{array}{l} \Rightarrow \frac{du = dx}{u = \underline{x} - 1} \\ \Leftrightarrow u + 1 = \underline{x} \end{array} \right\} = \frac{2}{7}(x-1)^{7/2} - \frac{2}{5}(x-1)^{5/2} + C.$$

Though the answer above is correct, one often factors the final answer:

$$= \frac{2}{35}(x-1)^{5/2}[5(x-1) - 7] + C = \frac{2}{35}(x-1)^{5/2}(5x-12) + C.$$

Example 6.4.11 Compute $\int \frac{x}{\sqrt{2x+1}} dx$.

Solution: We will work this problem twice using two different substitutions. The first is perhaps the most obvious, but the second has some appeal as well.

$$\int \frac{x}{\sqrt{2x+1}} dx = \int \frac{\frac{1}{2}(u-1)}{u^{1/2}} \cdot \frac{1}{2} du = \frac{1}{4} \int (u^{1/2} - u^{-1/2}) du$$

$$\left. \begin{array}{l} \Rightarrow \frac{du = 2 dx}{u = 2x + 1} \\ \Rightarrow \frac{1}{2} du = dx \\ \Rightarrow x = \frac{1}{2}(u-1) \end{array} \right\} = \frac{1}{4} \left(\frac{2}{3}u^{3/2} - 2u^{1/2} \right) + C = \frac{1}{6}(2x+1)^{3/2} - \frac{1}{2}(2x+1)^{1/2} + C.$$

Again one might factor, simplify and rearrange the variable parts of the answer to arrive at

$$= \frac{1}{6}(2x+1)^{1/2}[(2x+1) - 3] + C = \frac{1}{6}\sqrt{2x+1}(2x-2) + C = \frac{1}{3}(x-1)\sqrt{2x+1} + C.$$

For the alternative substitution we let $u = \sqrt{2x+1}$. Note how much of the integrand is then

absorbed into du (due to the relationship between the square root and its derivative).

$$\int \frac{x}{\sqrt{2x+1}} dx = \int \underbrace{\frac{1}{2}(u^2-1)}_x \underbrace{du}_{\frac{dx}{\sqrt{2x+1}}} = \frac{1}{2} \left[\frac{1}{3}u^3 - u \right] + C = \frac{1}{6}u^3 - \frac{1}{2}u + C$$

$$\left. \begin{array}{l} \Rightarrow u = \sqrt{2x+1} \\ \Rightarrow du = \frac{1}{2\sqrt{2x+1}} \cdot 2 dx \\ \Rightarrow du = \frac{1}{\sqrt{2x+1}} dx \\ \hline \Rightarrow u = \sqrt{2x+1} \\ \Rightarrow u^2 = 2x+1 \\ \Rightarrow \frac{1}{2}(u^2-1) = x \end{array} \right\} = \frac{1}{6} \left(\sqrt{2x+1} \right)^3 - \frac{1}{2}\sqrt{2x+1} + C \text{ (as before).}$$

It is important to notice that we used *the same* equation for u to calculate du , within a given strategy (though u , du were different for the different strategies). Also the reader should begin to see that we can make some rather interesting substitutions, so long as we are consistent when replacing every term inside the integral. In doing so, it will become apparent if (1) it is even possible to use a given substitution to rewrite the integral, and (2) even if so, is the new integral one which we can actually compute.

Example 6.4.12 Compute $\int \frac{x^3}{\sqrt{x^2+1}} dx$.

6.4.3 Other Miscellaneous Power Rule Substitutions

So far we have concentrated on algebraic (polynomial and rational-power), exponential and trigonometric functions in our substitution problems. It is also worth examining how power rules can arise from integrals involving logarithmic and arctrigonometric functions, which we do in this subsection.

Example 6.4.13 Compute $\int \frac{(\ln x)^5}{x} dx$.

Solution: Here we see a factor $\frac{1}{x}$, which is the derivative of $\ln x$, so the latter will be u :

$$\int \frac{(\ln x)^5}{x} dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{(\ln x)^6}{6} + C.$$

$$\left. \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right\}$$

Note that, as a general rule, if we have a function $f(x)$ with antiderivative $F(x)$, then we have¹⁷

$$\int \frac{f(\ln x)}{x} dx = \int f(u) du = F(u) + C = F(\ln x) + C. \quad (6.55)$$

$$\left. \begin{array}{l} u = \ln x \\ \Rightarrow du = \frac{1}{x} dx \end{array} \right\}$$

Similar formulas apply to the arc-trigonometric functions. Rather than list and commit to memorize them, it is better to look at the general idea that if, say, $\sin^{-1} x$ occurs in an integral, we would look immediately to see if its derivative, $\frac{1}{\sqrt{1-x^2}}$, also appears. Similarly for all functions.

Example 6.4.14 Compute $\int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx$.

Solution: Note here that if $u = \sin^{-1} x$ then our du below will account for $\frac{1}{\sqrt{1-x^2}} dx$:

$$\int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |\sin^{-1} x| + C.$$

$$\left. \begin{array}{l} u = \sin^{-1} x \\ \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx \end{array} \right\}$$

Example 6.4.15 Compute $\int \frac{\sec^{-1} x}{x\sqrt{x^2-1}} dx$. Assume $x > 0$ (or more precisely, $x \geq 1$).

Solution:

$$\int \frac{\sec^{-1} x}{x\sqrt{x^2-1}} dx = \int u du = \frac{1}{2}u^2 + C = \frac{(\sec^{-1} x)^2}{2} + C.$$

$$\left. \begin{array}{l} u = \sec^{-1} x \\ du = \frac{1}{x\sqrt{x^2-1}} dx \end{array} \right\}$$

In the example above, if instead $x < 0$ (actually $x < -1$), we would replace x by $-|x|$ in the denominator of the integrand, giving eventually $-\frac{1}{2}(\sec^{-1} x) + C$ for the antiderivative.

¹⁷In fact we can replace $\ln x$ with $\ln |x|$ in throughout the above.

Exercises

Compute the given indefinite integrals.

1. $\int \cos^3 x \sin x \, dx$
2. $\int \cos^3 2x \sin 2x \, dx$
3. $\int x^3(x^4 + 10) \, dx$
4. $\int [(2x + 1)(x^2 + x + 9)] \, dx$
5. $\int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$
6. $\int \frac{\ln x}{x} \, dx$
7. $\int \frac{1}{x \ln x} \, dx$
8. $\int \frac{x}{x^2 + 4} \, dx$
9. $\int \frac{(\tan^{-1} x)^2}{1 + x^2} \, dx$
10. $\int \cot x \, dx$
11. $\int \sec^2 x \tan x \, dx$
12. $\int \frac{1}{\sqrt{x}} \cdot (5 + \sqrt{x})^{10} \, dx$
13. $\int \frac{\cos^{-1} x}{\sqrt{1 - x^2}} \, dx$
14. Perform the following computation without rearrangement for both cases.
 - (a) $\int \sec x \cdot \sec x \tan x \, dx$
 - (b) $\int \tan x \sec^2 x \, dx$
 - (c) Explain why, though the answers “look” different, in fact are the same.
15. Without rearrangement, compute $\int \frac{\sec x \tan x}{\sec x} \, dx$.
16. Perform the previous computation after first simplifying the fraction (perhaps into sines and cosines).
17. Using the methods of Subsection 6.4.2, compute the following integral:

$$\int x\sqrt{1-x} \, dx.$$
18. Compute $\int x(x+5)^5 \, dx$
19. Compute $\int \frac{x}{\sqrt{2x+1}} \, dx$

6.5 Second Trigonometric Rules

We first looked at the simplest trigonometric integration rules—those arising from the derivatives of the trigonometric functions—in Section 6.1 (Subsection 6.1.4, page 409 to be more precise). Here we will complete the trigonometric rules in which one of the six basic trigonometric functions is the “outer” function. In fact we have four of the six antiderivatives we need: sine and cosine come quickly from the derivative formulas, and tangent and cotangent come from substitution arguments. As it turns out, secant and cosecant require a little more cleverness, and while we will not derive these from first principles, we will show that checking them is a quick and interesting derivative computation. Unfortunately (or fortunately, whatever your perspective) there are variations of the antiderivatives of tangent, cotangent, secant and cosecant. We will choose one form for each, but the well-informed student must be aware of the others to be prepared to discuss calculus topics among students with different backgrounds.¹⁸

6.5.1 Antiderivatives of the Six Trigonometric Functions

The antiderivatives of the six basic trigonometric functions are as follow:

$$\int \sin x \, dx = -\cos x + C, \quad (6.56)$$

$$\int \cos x \, dx = \sin x + C, \quad (6.57)$$

$$\int \tan x \, dx = \ln |\sec x| + C, \quad (6.58)$$

$$\int \cot x \, dx = -\ln |\csc x| + C, \quad (6.59)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C, \quad (6.60)$$

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C. \quad (6.61)$$

The first four of these can be verified mentally through quick derivative computations if the student is well enough versed in differentiation. The last two require some more care, but are

¹⁸In fact there are no strongly compelling reasons not to use

$$\int \tan x \, dx = -\ln |\cos x| + C,$$

$$\int \cot x \, dx = \ln |\sin x| + C,$$

which afterall have slightly simpler verifications by differentiation than (6.58) and (6.59). Here we have opted to use the latter, slightly more difficult formulas for a few reasons. First, they are themselves quite popular. Second, the reader used to (6.58) and (6.59) will be less likely to be confused when presented the simpler alternatives by a colleague (or future professor) with a different background, while the reader used to those simpler alternatives may have some initial difficulty if similarly presented our forms here. Finally, there is so much added structure, both calculus and algebraic, found in the context of the secant and cosecant functions so it is important to be familiar and comfortable with them.

Admittedly, however, if (6.58) and (6.59) were not so common we would likely opt for the simpler forms.

somewhat interesting to check. For instance, we can verify (6.60) as follows:

$$\begin{aligned} \frac{d \ln |\sec x + \tan x|}{dx} &= \frac{1}{\sec x + \tan x} \cdot \frac{d(\sec x + \tan x)}{dx} \\ &= \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) \\ &= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x, \text{ q.e.d.} \end{aligned}$$

It is not entirely obvious how one would derive antiderivatives of the secant and cosecant functions, and so it is important to memorize those especially. Indeed it is likely these were discovered through experimentation, and such results are often very time consuming to reproduce from first principles if one has to re-invent “the trick,” one of which will be explored in the exercises. In fact we will later show a popular alternative antiderivative for the cosecant, and a not-so-popular alternative for the secant. The alternatives for the tangent and cotangent are similar in popularity to those we will use for our standards.

There is little we can do with just (6.56)–(6.61) as they stand, but we nonetheless explore a few examples quickly.

Example 6.5.1 *Below are two quick antiderivative computations involving our basic trigonometric integral formulas.*

- $\int \frac{\sin^2 x + \cos x}{\sin x} dx = \int (\sin x + \cot x) dx = -\cos x - \ln |\csc x| + C.$
- $\int (x + \sec x) dx = \frac{x^2}{2} + \ln |\sec x + \tan x| + C.$

Example 6.5.2 *Suppose $v(t) = 1 + \tan t$, and $s(\pi/3) = 7$. Find $s(t)$, and the range of t for which the solution is valid.*

Solution: We know that $s(t)$ is an antiderivative of $v(t)$, so we write the following, realizing that we will use our one datum ($s(\pi/3) = 7$) to find the additive constant later.

$$s(t) = \int v(t) dt = \int (1 + \tan t) dt = t + \ln |\sec t| + C.$$

So far $s(t) = t + \ln |\sec t| + C$, and $s(\pi/3) = 7$, so

$$7 = \frac{\pi}{3} + \ln \left| \sec \frac{\pi}{3} \right| + C \iff 7 = \frac{\pi}{3} + \ln 2 + C \iff 7 = \frac{\pi}{3} + \ln 2 + C,$$

and so $C = 7 - \frac{\pi}{3} - \ln 2$. Hence

$$v(t) = t + \ln |\sec t| + 7 - \frac{\pi}{3} - \ln 2.$$

6.5.2 Substitution and the Basic Trigonometric Functions, Part I

A student who uses calculus extensively is likely to eventually encounter an antiderivative problem where the form is particularly difficult or obscure, in which case it is common to refer to so-called tables of integrals. These usually contain all the basic forms as well as those which

would be difficult enough to warrant a search through such a reference.¹⁹ It is interesting to note that most modern tables of integrals do not use the common variable x in the formulas, but instead use u , which is the most common variable for substitution type problems. This is because substitution is so ubiquitous that it is assumed the reader might not need a form exactly as it is in the table, but rather needs one which becomes one of the forms (or a constant multiple of one of the forms) found in the table only after a substitution. In that spirit, the standard method of listing the antiderivatives of the basic six trigonometric functions is as follows:

$$\int \sin u \, du = -\cos u + C, \quad (6.62)$$

$$\int \cos u \, du = \sin u + C, \quad (6.63)$$

$$\int \tan u \, du = \ln |\sec u| + C, \quad (6.64)$$

$$\int \cot u \, du = -\ln |\csc u| + C, \quad (6.65)$$

$$\int \sec u \, du = \ln |\sec u + \tan u| + C, \quad (6.66)$$

$$\int \csc u \, du = -\ln |\csc u + \cot u| + C. \quad (6.67)$$

Of course these are just our previous formulas (6.56)–(6.61), from page 431, but with the entire integral written in the variable u instead of x . However, each of these properly interpreted contains a reverse chain rule, also known as a substitution-type form. So for instance, if $u = u(x)$, then $du = u'(x) \, dx$ and so we can read (6.64) as

$$\int \underbrace{\tan u(x)}_{\tan u} \underbrace{u'(x) \, dx}_{du} = \ln |\sec u(x)| + C,$$

verified by differentiation:

$$\frac{d}{dx} \ln |\sec u(x)| = \frac{1}{\sec u(x)} \cdot \frac{d \sec u(x)}{dx} = \frac{1}{\sec u(x)} \cdot \sec u(x) \tan u(x) \cdot \frac{du(x)}{dx} = \tan u(x) \cdot u'(x),$$

q.e.d. So forms (6.62)–(6.67) are all forms in which a basic trigonometric function of some function $u(x)$, and the derivative $u'(x)$, and the differential dx are the nonconstant factors of the integral. We now look at several examples.

Example 6.5.3 Compute $\int x \sin x^2 \, dx$.

¹⁹In a typical calculus class, the professor usually has to answer the question of why students have to learn all of the difficult integration techniques when there are references available. The answer is several-fold, and we make just a few points addressing it here.

First, many problems are simple enough to not require reference, and using tables for every problem becomes akin to looking up every word in a dictionary as one reads a newspaper, for instance. Second, it is not even possible to use tables for every integration problem simply because many problems which can be accomplished through the techniques will not match what is in the tables. Third, on occasion there will be a technicality which the editors of the tables did not anticipate for a particular problem, or a mistake they did not catch, and so reliance on tables can be problematic. (On this last point, the same is true of mathematical software packages which claim to compute integrals.)

We will eventually make use of tables to a limited extent, to see what technicalities need to be addressed when using them, and to give some flavor of what kinds of integrals one can find in standard lists. Indeed, every serious calculus student should eventually possess such a reference, though again it should be used sparingly.

Solution: As often occurs, the form is not exact but a constant multiple of one of our forms, this time (6.62), and furthermore the order of the factors is changed. Here we see that the factor x is a constant multiple of $u'(x)$ if $u(x) = x^2$, so the extra factor of x can be “absorbed” in the differential du after the substitution. This ultimately leaves us with the problem of finding the antiderivative of a sine function.

$$\int \underbrace{x \sin x^2}_{\underline{dx}} = \int \sin u \cdot \underbrace{\frac{1}{2} du}_{\underline{du}} = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos x^2 + C.$$

$$\begin{aligned} & \left. \begin{aligned} \Rightarrow \quad & \left. \begin{aligned} u &= x^2 \\ du &= 2x \underline{dx} \end{aligned} \right\} \\ \Leftrightarrow \quad & \frac{1}{2} du = \underline{x dx} \end{aligned} \right\} \end{aligned}$$

The above example can be quickly checked by differentiation.

Example 6.5.4 Compute $\int e^x \cot e^x dx$.

Solution: Here we see the derivative of the argument e^x of the cotangent function is also present as a multiplicative factor.

$$\int \underbrace{e^x \cot e^x}_{\underline{dx}} = \int \cot u \underline{du} = -\ln |\csc u| + C = -\ln |\csc e^x| + C.$$

$$\begin{aligned} & \left. \begin{aligned} \Rightarrow \quad & \left. \begin{aligned} u &= e^x \\ du &= \underline{e^x dx} \end{aligned} \right\} \end{aligned} \right\} \end{aligned}$$

Example 6.5.5 Compute $\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx$.

Solution: Here the factor $\frac{1}{\sqrt{x}}$ is in fact a constant multiple of the derivative of \sqrt{x} , the argument of the trigonometric function. Thus we take $u = \sqrt{x}$, and then the resulting du will absorb the $\frac{1}{\sqrt{x}}$ term:

$$\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx = \int \sec \sqrt{x} \cdot \underbrace{\frac{1}{\sqrt{x}} dx}_{\underline{du}} = \int \sec u \cdot 2 \underline{du} = 2 \ln |\sec u + \tan u| + C$$

$$\begin{aligned} & \left. \begin{aligned} \Rightarrow \quad & \left. \begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}} dx \end{aligned} \right\} \\ \Leftrightarrow \quad & 2 du = \underline{\frac{1}{\sqrt{x}} dx} \end{aligned} \right\} = 2 \ln |\sec \sqrt{x} + \tan \sqrt{x}| + C. \end{aligned}$$

Example 6.5.6 Compute $\int \frac{\cos(1 + 4 \ln x)}{x} dx$.

Solution: Here we see the derivative of $(1 + 4 \ln x)$ appearing as a factor as well, except for

a constant factor.

$$\int \frac{\cos(1 + 4 \ln x)}{x} dx = \int \cos(1 + 4 \ln x) \cdot \underbrace{\frac{1}{x} dx}_{\frac{1}{4} du} = \int \cos u \cdot \underbrace{\frac{1}{4} du}_{\frac{1}{4} du} = \frac{1}{4} \sin u + C$$

$$\begin{aligned} \implies & \left. \begin{aligned} u &= 1 + 4 \ln x \\ du &= 4 \cdot \frac{1}{x} dx \end{aligned} \right\} = \frac{1}{4} \sin(1 + 4 \ln x) + C. \\ \iff & \frac{1}{4} \cdot du = \frac{1}{x} dx \end{aligned}$$

Example 6.5.7 Compute $\int x^2 \csc(\cos x^3) \sin x^3 dx$.

Solution: To be clear, first we note that the integrand is the product of three factors:

$$x^2 \cdot \csc(\cos x^3) \cdot \sin x^3,$$

so the argument of the cosecant is $\cos x^3$. Now we will compute this two different ways. The first method requires two substitutions, which is an option that students must be aware is legitimate, assuming all computations are made carefully and consistently.

Method 1. Here we will first make a substitution $u = x^3$ to yield a simpler integral without any polynomial factors, though our new integral will still require some work.

$$\int \underbrace{x^2}_{\frac{1}{3} du} \csc(\cos x^3) \sin x^3 \underbrace{dx}_{du} = \int \csc(\cos u) \sin u \cdot \frac{1}{3} du$$

$$\begin{aligned} \implies & \left. \begin{aligned} u &= x^3 \\ du &= 3x^2 dx \end{aligned} \right\} \\ \iff & \frac{1}{3} du = x^2 dx \end{aligned}$$

So at this point our problem reduces to computing $\int \csc(\cos u) \sin u \cdot \frac{1}{3} du$. To do so we use another substitution, noting that $\sin u$ is the derivative—up to a multiplicative constant—of $\cos u$ (with respect to u this time). To remain consistent this second substitution must use a new variable (lest we give one letter two different meanings within the same problem, which would be contradictory!). So we call our new variable something other than u or x . A commonly used variable at this stage is w :

$$\frac{1}{3} \int \csc(\cos u) \underbrace{\sin u du}_{dw} = \frac{1}{3} \int \csc w \underbrace{(-1) dw}_{dw} = -\frac{1}{3} [-\ln |\csc w + \cot w|] + C$$

$$\begin{aligned} \implies & \left. \begin{aligned} w &= \cos u \\ dw &= -\sin u du \end{aligned} \right\} = \frac{1}{3} \ln |\csc w + \cot w| + C \\ \iff & (-1)dw = \sin u du \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \ln |\csc(\cos u) + \cot(\cos u)| + C \\ &= \frac{1}{3} \ln |\csc(\cos x^3) + \cot(\cos x^3)| + C. \end{aligned}$$

Note how we computed the antiderivative in w , which we then replaced by its expression in u , and finally by the definition of u in terms of x .

Method 2. If we can see far enough ahead, we can combine both substitutions into one. For clarity we will use a different variable—namely z —here (though by convention one would usually use u). We choose $z = \cos x^3$, noting that its derivative, requiring the chain rule, will have a $\sin x^3$ and a x^2 term (ignoring multiplicative constants), which leaves us with a constant multiple of $\int \csc z \, dz$, for which we have a formula.

$$\int \underbrace{x^2}_{\text{constant}} \csc(\cos x^3) \underbrace{\sin x^3 \, dx}_{\text{chain rule}} = \int \csc z \cdot \underbrace{\frac{-1}{3}}_{\text{chain rule}} dz = -\frac{1}{3} \cdot [-\ln |\csc z + \cot z|] + C$$

$$\begin{aligned} \implies & \left. \begin{aligned} z &= \cos x^3 \\ dz &= \underbrace{-\sin x^3}_{\text{chain rule}} \cdot \underbrace{3x^2 \, dx}_{\text{chain rule}} \end{aligned} \right\} = \frac{1}{3} \ln |\csc(\cos x^3) + \cot(\cos x^3)| + C. \\ \iff & -\frac{1}{3} dz = \underbrace{x^2 \sin x^3 \, dx}_{\text{chain rule}} \end{aligned}$$

In the previous example, the second method in fact just combines the two substitutions from the first method into one. Indeed, the formula for w in terms of x is the same as that of z .²⁰

$$w = \csc(u) = \csc(\cos x^3) = z.$$

When one is well practiced in substitution the second method will likely be chosen. However, it is important also for the student to realize that even if a substitution does not achieve an integral that can be immediately computed, that does not mean that the particular substitution need be abandoned. If the new integral is simpler, then the first substitution can be worthwhile. In fact in later sections we will on occasion *require* multiple substitutions. Of course it is important that all steps be carried out carefully, accurately and consistently.

In this section we concentrated on those integrals which reduce to integrals of a single trigonometric function, perhaps with the aid of a substitution. In Chapter 7 (and more specifically Section 7.3) we will look at the many techniques for computing those integrals which contain several factors of trigonometric functions, and no other factors. The techniques of our present section will be called upon often, but these are only a small part of the needed knowledge for computing the “trigonometric integrals” of the later sections. But in fact we have some other techniques already. For instance, there were the first trigonometric integral formulas we had in Section 6.1, Subsection 6.1.4 which arose from the derivative rules for the six basic trigonometric functions. In fact we had one other technique for dealing with some trigonometric integrals, which was substitution in the case we could rewrite the trigonometric integral as a power-rule type integral.²¹

²⁰Similarly, though perhaps not so obviously, when we recall these variables are all functions of x we also have $dw = dz$:

$$\begin{aligned} dw &= \frac{dw}{du} \cdot \frac{du}{dx} \cdot dx \\ &= (-\sin u) \cdot (3x^2) \, dx \\ &= -\sin x^3 \cdot 3x^2 \, dx \\ &= dz. \end{aligned}$$

Again we see the power of the Leibniz notation in what is essentially a chain rule. Of course we should expect that $w = z \implies dw = dz$. But also we see that while there are obvious algebraic consistencies in our substitution method, there are also consequent calculus consistencies which, while more subtle, are still correct when we perform all the computations correctly.

²¹In fact there will be several other substitution type arguments we will make for trigonometric integrals besides those which yield power rules.

Example 6.5.8 Compute $\int \frac{\sin 3x}{\cos^5 3x} dx$.

Solution: Here we see that we have the derivative of the cosine function is present as a factor, and we are left with a power of the cosine:

$$\int \frac{\sin 3x}{\cos^5 3x} dx = \int (\cos 3x)^{-5} \sin 3x dx = \int u^{-5} \cdot \frac{-1}{3} du$$

$$\begin{aligned} \Rightarrow \left. \begin{array}{l} u = \cos 3x \\ du = -\sin 3x \cdot 3 dx \\ \Leftrightarrow \frac{-1}{3} du = \sin 3x dx \end{array} \right\} &= -\frac{1}{3} \cdot \frac{-1}{4} u^{-4} + C \\ &= \frac{1}{12} \cos^{-4} 3x + C \\ &= \frac{1}{12} \sec^4 3x + C. \end{aligned}$$

The integration techniques we encounter throughout the book are many and varied. We will see later how a slight change in a problem can substantially change the result, its difficulty, or the technique used to achieve it. We have seen this phenomenon before. Consider for instance

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) + C,$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C,$$

$$\int \frac{x}{\sqrt{1 - x^2}} dx = -\sqrt{1 - x^2} + C,$$

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C,$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C,$$

$$\int \sec^2 x dx = \tan x + C,$$

$$\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln |\sec x + \tan x|) + C.$$

In fact this last problem will have to wait until Chapter 7, and is quite long and technical. Even the verification by differentiation is nontrivial, and requires one to employ some trigonometric identity along the way. Suffice for now to simply note that the techniques, and results, for even these first three powers of the secant are all very different.

Computing antiderivatives in good time requires the ability to recognize which technique will work for a particular problem. That in turn requires a fairly complete knowledge of the techniques, even to the extent that one can anticipate the outcomes of several later steps. Of course practice is one key to gaining this understanding. For that reason this chapter will contain one section in which the exercises' required techniques are purposely randomized, by method as well as difficulty.

Exercises

1. By differentiation, verify each of our basic six trigonometric integrals in the forms we use, (6.56)–(6.61). For reference see the proof for the secant, page 432.

$$(a) \int \sin x \, dx = -\cos x + C$$

$$(b) \int \cos x \, dx = \sin x + C$$

$$(c) \int \tan x \, dx = \ln |\sec x| + C$$

$$(d) \int \cot x \, dx = -\ln |\csc x| + C$$

$$(e) \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$(f) \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

2. Compute the following integrals.

$$(a) \int x \sec(x^2 + 1) \, dx$$

$$(b) \int \frac{\tan(\ln x)}{x} \, dx$$

$$(c) \int \frac{\sin(\frac{1}{x})}{x^2} \, dx$$

$$(d) \int \sqrt{x} \csc(x\sqrt{x}) \, dx$$

$$(e) \int x^3 e^{5x^4} \cot(6e^{5x^4}) \, dx$$

$$(f) \int \frac{x}{\sin 3x^2} \, dx$$

$$(g) \int 2^x \cos(3 \cdot 2^x) \, dx$$

$$(h) \int (1 - \sec x)^2 \, dx$$

$$(i) \int (\tan 7x - 2)^2 \, dx$$

$$(j) \int \frac{1 - \sin^2 x}{\cos x} \, dx$$

$$(k) \int \frac{\sin^2 x}{\cos x} \, dx \text{ (for fun)}$$

$$(l) \int \frac{(1 - \cos x)^2}{\sin x} \, dx \text{ (for fun)}$$

3. Derive our formula (above) for the integral of $\sec x$ by the following algebraic device, namely multiplying and dividing by $\sec x + \tan x$ within the integral, i.e.,

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx,$$

and then using an appropriate substitution argument.

4. By differentiation, verify each of the following alternative integration formulas.

$$(a) \int \tan x \, dx = -\ln |\cos x| + C.$$

$$(b) \int \cot x \, dx = \ln |\sin x| + C.$$

$$(c) \int \sec x \, dx = -\ln |\sec x - \tan x| + C.$$

$$(d) \int \csc x \, dx = \ln |\csc x - \cot x| + C.$$

6.6 Substitution with All Basic Forms

In this section we will add to our forms for substitution and recall some rather general guidelines for substitution. Except for our four new trigonometric forms from Section 6.5, all forms in this chapter derive directly from derivative rules. These comprise what we call here the *basic* integration rules. Each is based upon a single function specific to the rule. So for instance, we will have in our list the following:

$$\int \frac{1}{u^2 + 1} du = \tan^{-1} u + C.$$

As before, the usual variable of integration in the given problem will likely be x , but the *form* may be ultimately as above, except for multiplicative constants, where $u = u(x)$ and then $du = u'(x) dx$ contains another factor from the original integral. So for instance we might see

$$\begin{aligned} \int \frac{x}{x^4 + 1} dx &= \int \frac{1}{(x^2)^2 + 1} \cdot \underbrace{x dx} = \int \frac{1}{u^2 + 1} \cdot \underbrace{\frac{1}{2} du} = \frac{1}{2} \tan^{-1} u + C \\ \Rightarrow \left. \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right\} &= \frac{1}{2} \tan^{-1} x^2 + C. \\ \Leftrightarrow \left. \frac{1}{2} dx = \underbrace{x dx} \right\} & \end{aligned}$$

One clue that we might try $u = x^2$ was that its derivative was a factor in the integrand in the form of the factor x , again excepting multiplicative constants, and so we wrote that factor separately next to the differential dx . As it turned out, the rest of the integrand could indeed be written as a function of $u = x^2$.

Reading the problem above backwards, the arctangent is the “outer function” of a chain rule differentiation problem, and x^2 was the “inner function.” Put in terms of integration, the *form* was $\int \frac{1}{u^2+1} du$, excepting multiplicative constants, with the “inner function” $u = x^2$. The arctangent appeared because of the ultimate form of the integral, in terms of $u = x^2$.

But note that the arctangent can also appear as the “inner function,” which we may wish to set equal to u . So for instance,

$$\begin{aligned} \int \frac{(\tan^{-1} x)^2}{x^2 + 1} dx &= \int (\tan^{-1} x)^2 \cdot \underbrace{\frac{1}{x^2 + 1} dx} = \int u^2 \underbrace{du} = \frac{u^3}{3} + C \\ \Rightarrow \left. \begin{array}{l} u = \tan^{-1} x \\ du = \frac{1}{x^2 + 1} dx \end{array} \right\} &= \frac{(\tan^{-1} x)^3}{3} + C. \end{aligned}$$

In all these cases, we are looking for some $u = u(x)$ so that

- one major (nonconstant) factor of the integral can be simply written $f(u)$, i.e., u is an “inner function” of some composite function $f(u(x))$ which appears in the integrand,
- the remaining factors of the integrand will collectively be a constant multiple of $du = u'(x) dx$,
- and finally, so that $\int f(u) du$ is an integral we can compute, i.e., we know the antiderivative of f .

So of course identifying u is the key, and in doing so we have to be sure its derivative is also present, and finally that we are left with an integral—albeit in u —which we can handle.²²

6.6.1 List of Basic Forms

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1, \quad (6.68)$$

$$\int \frac{1}{u} du = \ln |u| + C, \quad (6.69)$$

$$\int \sin u du = -\cos u + C, \quad (6.70)$$

$$\int \cos u du = \sin u + C, \quad (6.71)$$

$$\int \tan u du = \ln |\sec u| + C, \quad (6.72)$$

$$\int \cot u du = -\ln |\csc u| + C, \quad (6.73)$$

$$\int \sec u du = \ln |\sec u + \tan u| + C, \quad (6.74)$$

$$\int \csc u du = -\ln |\csc u + \cot u| + C, \quad (6.75)$$

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C, \quad (6.76)$$

$$\int \frac{1}{u^2+1} du = \tan^{-1} u + C, \quad (6.77)$$

$$\int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1} |u| + C, \quad (6.78)$$

$$\int e^u du = e^u + C, \quad (6.79)$$

$$\int a^u du = \frac{a^u}{\ln a} + C, \quad (6.80)$$

$$\int \sec^2 u du = \tan u + C, \quad (6.81)$$

$$\int \csc^2 u du = -\cot u + C, \quad (6.82)$$

$$\int \sec u \tan u du = \sec u + C, \quad (6.83)$$

$$\int \csc u \cot u du = -\csc u + C. \quad (6.84)$$

²²This all assumes that there is a substitution which will make the integral into one of these simple forms. It is not always the case. One which occurs in probability and other subjects is $\int e^{x^2} dx$, which can not be changed by substitution into a useful form for simple integration. In fact it will not succumb to any of the methods in this or the next chapter. We will eventually find a way to deal with this integral, in Chapter 11. In the meantime it is actually a good exercise to see why this integral can not be forced into any of our methods. Indeed, seeing what goes wrong in such a case very well complements seeing what goes right in the cases where substitution, and later methods, do achieve an answer.

It can not be stressed too much that each form given assumes that a substitution may be required. So again, the following formulas say the same:

$$\int e^u du = e^u + C, \quad \int e^{u(x)} u'(x) dx = \int e^{u(x)} \cdot \frac{du(x)}{dx} dx = e^{u(x)} + C.$$

Recognizing when we have such a form is again key to using these formulas.

Example 6.6.1 Compute $\int x e^{x^2} dx$.

Solution: Here we see the derivative of x^2 appearing as the factor x , except for a constant multiple. Hence we let $u = x^2$, and the du will contain the other factor x , leaving an integral in one of our standard forms, namely (6.79), and nothing else except a multiplicative constant.

$$\begin{aligned} \int x e^{x^2} dx &= \int e^u \cdot \frac{1}{2} du = \frac{1}{2} e^u + C \\ \implies \left. \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right\} &= \frac{1}{2} e^{x^2} + C. \\ \iff \frac{1}{2} du = x dx & \end{aligned}$$

Example 6.6.2 Compute $\int \frac{x^3}{\sqrt{1-x^8}} dx$.

Solution: At first it is tempting to let $u = 1 - x^8$ and hope this will become a power rule, except that such u implies $du = -8x^7$, which is very different from a constant multiple of the other factor here, namely x^3 .

In fact, the other factor can be a good source of information about what to set equal to u . Indeed the factor x^3 will be part of the differential of $u = x^4$, and then we can recognize a form which will yield an arcsine ultimately, i.e., form (6.76).

$$\begin{aligned} \int \frac{x^3}{\sqrt{1-x^8}} dx &= \int \frac{1}{\sqrt{1-(x^4)^2}} \cdot x^3 dx = \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{4} du = \frac{1}{4} \sin^{-1} u + C \\ \implies \left. \begin{array}{l} u = x^4 \\ du = 4x^3 dx \end{array} \right\} &= \frac{1}{4} \sin^{-1} (x^4) + C. \\ \iff \frac{1}{4} du = x^3 dx & \end{aligned}$$

As with the power rule, there are occasions where the derivative of our u is a nonzero constant, and thus a constant multiple of every other nonzero constant. While these integrals are arguably easier than the others we encounter here, their relative simplicity can be a source of confusion.

Example 6.6.3 Compute $\int \csc 5x dx$.

Solution: Here we simply let $u = 5x$.

$$\begin{aligned} \int \csc 5x dx &= \int \csc u \cdot \frac{1}{5} du = -\frac{1}{5} \ln |\csc u + \cot u| + C \\ \implies \left. \begin{array}{l} u = 5x \\ du = d dx \end{array} \right\} &= -\frac{1}{5} \ln |\csc 5x + \cot 5x| + C. \\ \iff \frac{1}{5} du = dx & \end{aligned}$$

Example 6.6.4 Compute $\int 2^{3^x} \cdot 3^x dx$.

Solution: Here we will need (6.80) eventually, but first we simply notice that the factor 3^x is a constant multiple of the exponent in the first factor, so we let $u = 3^x$.

$$\int 2^{3^x} \cdot \underbrace{3^x dx}_{\ln 3} = \int 2^u \cdot \frac{1}{\ln 3} dx = \frac{1}{\ln 3} \cdot \frac{1}{\ln 2} \cdot 2^u + C$$

$$\Rightarrow \left. \begin{array}{l} u = 3^x \\ du = \underbrace{3^x \ln 3 dx} \end{array} \right\} = \frac{2^{3^x}}{\ln 3 \cdot \ln 2} + C.$$

$$\Leftrightarrow \frac{1}{\ln 3} du = \underbrace{3^x dx}$$

The example above shows the importance of following the various constant factors through the integration. Students who rely upon guessing the answers, without performing the formal substitution steps, are much more likely to misplace one or more constant factors.

For further practice we consider more basic examples.

Example 6.6.5 Compute $\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx$.

Solution: The key here is that the derivative of the argument of the secant is also present as a factor. Recall $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$, which is obvious when the radicals are written as $1/2$ -powers.

$$\int \frac{\sec \sqrt{x}}{\sqrt{x}} dx = \int \sec \sqrt{x} \cdot \frac{1}{\sqrt{x}} dx = \int \sec u \cdot \underbrace{2 du} = 2 \ln |\sec u + \tan u| + C$$

$$\Rightarrow \left. \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2} \cdot \underbrace{x^{-1/2} dx} \end{array} \right\} = 2 \ln |\sec \sqrt{x} + \tan \sqrt{x}| + C.$$

In the next section we will further explore the arctrigonometric antiderivatives by considering further complications. For now we only look at two such complications, first involving the arctangent (though the arcsine has a similar potential complication), and then the arcsecant, which has the same complications as the others and then one more.

Example 6.6.6 Compute $\int \frac{x^2}{1 + 25x^6} dx$.

Solution: There are two clues directing our choice of u . First, we see the factor x^2 , which is a multiple of the derivative of x^3 . Then we see that the denominator can be written as $1 + (5x^3)^2$, which we can put into the form yielding the arctangent, namely (6.77).

$$\int \frac{x^2}{1 + 25x^6} dx = \int \frac{1}{1 + (5x^3)^2} \cdot \underbrace{x^2 dx} = \int \frac{1}{1 + u^2} \cdot \frac{1}{\underbrace{15}} du = \frac{1}{15} \tan^{-1} u + C$$

$$\Rightarrow \left. \begin{array}{l} u = 5x^3 \\ du = 15 \underbrace{x^2 dx} \end{array} \right\} = \frac{1}{15} \tan^{-1} (5x^3) + C.$$

$$\Leftrightarrow \frac{1}{15} du = \underbrace{x^2 dx}$$

The complication in this example is fairly benign: that the u term contains a multiplicative constant. Here we wanted $25x^6$ to be u^2 for the form (6.77), so we took $u = 5x^3$.²³ Fortunately

²³Note that we could also have used $u = -5x^3$, but then $du = -15x^2 dx$, and so our answer would ultimately be (as the reader should verify) $-\frac{1}{15} \tan^{-1}(-5x^3) + C$. In fact that is the same as the answer we got, since the arctangent is an “odd” function, that is, $\tan^{-1}(-z) = -\tan^{-1} z$.

this was consistent with the du containing the x^2 term of the integrand, and that form (6.77) could actually be used. In the next section we will see how to deal with cases where the additive constant in the denominator of the integrand is not 1. For now we look at another complication which is somewhat specific to the arcsecant form.

Example 6.6.7 Compute $\int \frac{1}{x\sqrt{9x^2-1}} dx$.

Solution: Because a new complication needs to be explained while the problem is solved, the organization will be slightly different than previous exercises, but every technique used below has appeared previously. Note that we are trying to fit this integral into form (6.78).

Here we want $u^2 = 9x^2$ so we have $\sqrt{u^2-1}$ as one factor in the denominator of our integrand. Thus will let $u = 3x \implies du = 3 dx \iff \frac{1}{3} du = dx$. Our integral so far is then

$$\int \frac{1}{x\sqrt{u^2-1}} \cdot \frac{1}{3} du.$$

Now, all of our integral formulas require just one variable, but in fact the integral above makes sense because of the relationship between x and u . But to use a formula we have to put it all into the new variable, namely u . So there is one term left, which is the factor x on the bottom, which has to be put into u -terms. For that we go back to our original substitution and note that $u = 3x \iff x = \frac{1}{3}u$. Now we continue:

$$\begin{aligned} \int \frac{1}{x\sqrt{9x^2-1}} dx &= \int \frac{1}{x\sqrt{u^2-1}} \cdot \frac{1}{3} du \\ &= \int \frac{1}{\frac{1}{3}u\sqrt{u^2-1}} \cdot \frac{1}{3} du \\ &= \int \frac{1}{u\sqrt{u^2-1}} du \\ &= \sec^{-1}|u| + C \\ &= \sec^{-1}|3x| + C. \end{aligned}$$

With practice, our process of solving the above problem could more resemble the following:

$$\begin{aligned} \int \frac{1}{x\sqrt{9x^2-1}} dx &= \int \frac{1}{\tilde{x}\sqrt{(3\tilde{x})^2-1}} \frac{d\tilde{x}}{\tilde{x}} = \int \frac{1}{\frac{1}{3}u\sqrt{u^2-1}} \cdot \frac{1}{3} du \\ \left. \begin{aligned} &u = 3x \\ \implies du &= 3 \underline{d\tilde{x}} \\ \implies \frac{1}{3} du &= \underline{d\tilde{x}}, \\ \text{Also, } \frac{1}{3} du &= \underline{d\tilde{x}}, \\ \implies \frac{1}{3} du &= \underline{d\tilde{x}}, \\ \iff \frac{1}{3} u &= \tilde{x} \end{aligned} \right\} &= \int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1}|u| + C = \sec^{-1}|3x| + C. \end{aligned}$$

The next example illustrates why it is important to recall that $(a^n)^m = a^{m \cdot n}$. In particular for the example below, $e^{2x} = (e^x)^2$.

Example 6.6.8 Compute $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$.

Solution: Here we note a form $\int \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \cdot dx$ if we let $u = e^x$.

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}} dx &= \int \frac{1}{\sqrt{1-e^{2x}}} \cdot \underbrace{e^x dx}_{u = e^x} \\ &= \left. \begin{array}{l} u = e^x \\ \Rightarrow du = \underbrace{e^x dx} \end{array} \right\} = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1} e^x + C. \end{aligned}$$

On the other hand, we have to be careful that we do not try to “read” arc-trigonometric or other forms into what may be simple power rules.

Example 6.6.9 Compute $\int \frac{e^{2x}}{\sqrt{1-e^{2x}}} dx$.

Solution Here we note that the derivative of the function under the radical is also a factor, leaving only constant multiple factors:

$$\begin{aligned} \int \frac{e^{2x}}{\sqrt{1-e^{2x}}} dx &= \int (1-e^{2x})^{-1/2} \cdot \underbrace{e^{2x} dx}_{u = 1-e^{2x}} = \int u^{-1/2} \cdot \underbrace{\frac{-1}{2} du}_{\Rightarrow du = -e^{2x} \cdot 2 dx} \\ &= \left. \begin{array}{l} u = 1 - e^{2x} \\ \Rightarrow du = -e^{2x} \cdot 2 dx \\ \Rightarrow \frac{-1}{2} du = \underbrace{e^{2x} dx} \end{array} \right\} = \frac{-1}{2} \cdot 2u^{1/2} + C = -\sqrt{1-e^{2x}} + C. \end{aligned}$$

Exercises

1. Compute $\int \frac{1}{(x^2 + 1) \tan^{-1} x} dx$
2. Compute $\int \frac{1 + e^x}{x + e^x} dx$
3. Compute $\int \frac{\sec \ln x}{x} dx$
4. Compute $\int \frac{\sqrt[3]{\ln 9x}}{x} dx$
5. Compute $\int \frac{x^2}{x^3 + 1} dx$
6. Compute $\int \frac{x^3 + 1}{x^2} dx$
7. Compute $\int \frac{x^2}{x^6 + 1} dx$
8. Compute $\int \frac{\sec x \tan x}{\sqrt{1 - \sec x}} dx$
9. Compute $\int \frac{e^x}{(1 + e^x)^2} dx$
10. Compute $\int \csc^2 3x \cot^6 3x dx$
11. Compute $\int \frac{1}{\sqrt{x}(x + 1)} dx$
12. Compute $\int 2^x 3^{2^x} dx$
13. Compute $\int \frac{2^x}{1 + 4^x} dx$
14. Compute $\int 5^x \sec 5^x dx$
15. Compute $\int \frac{1}{\sqrt{1 - 5x^2}} dx$
16. Compute $\int \frac{x - 1}{\sqrt{1 - 4x^2}} dx$
17. Compute $\int x e^{x^2} dx$ two ways.
 - (a) Let $u = x^2$ (as in Example 6.6.1).
 - (b) Instead let $u = e^{x^2}$.
18. In the spirit of the previous exercise, compute $\int \frac{e^{1/x}}{x^2} dx$ two ways, i.e., using two different substitutions.
19. Compute $\int \sec^2 x \tan x dx$ two different ways.
 - (a) Let $u = \tan x$.
 - (b) Instead let $u = \sec x$.
 - (c) Explain why the two answers are equivalent.
20. Though it can not be rewritten into a basic form, it is interesting to attempt some substitutions for $\int e^{x^2} dx$.
 - (a) Let $u = x^2$ and attempt to rewrite the integral with this substitution.
 - (b) Let $u = x$ and attempt to rewrite the integral with this substitution. Explain why this is never useful for a substitution attempt for any integral of type $\int f(x) dx$.

6.7 Further Arctrigonometric Forms

Here we will still use the same arctrigonometric forms we had before, namely

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C, \quad (6.85)$$

$$\int \frac{1}{u^2+1} du = \tan^{-1} u + C, \quad (6.86)$$

$$\int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1} |u| + C. \quad (6.87)$$

What is different in this section is that our integrals will need to be algebraically rewritten into these forms, and this will require more than the previous substitution.

Each of the integrals in (6.85), (6.86) and (6.87) need to be exactly as they are stated. For instance, replacing $1-u^2$ with $1+u^2$ or u^2-1 in (6.85) will give completely different antiderivatives. In fact, even the domain of the integrand would be completely different with any such changes! Similar changes would substantially alter the results in (6.86) and (6.87).

In this section we will have integrands which we can algebraically rewrite so they conform to one of the forms (6.85), (6.86) or (6.87). In fact there are only a couple of algebraic “tricks” which we introduce here. The first of these is to force the denominators to have the additive constant 1, where originally there may be another constant. This is accomplished through simple factoring techniques. The second technique is “completing the square,” where appropriate, and then using the first technique to finish rewriting the integrand. With substitution there will often be further multiplicative constants to accommodate as well.

6.7.1 Factoring to Achieve “1”

Each of our integrals (6.85), (6.86) and (6.87) have the number 1 conspicuously appearing in the denominator, near the u^2 term. Any other nonzero *constant* there will have an effect on the vertical and horizontal scaling of the function in ways we can not ignore in the formula. To compensate is fairly straightforward: factor the constant, and see what should be called “ u^2 .”

Example 6.7.1 Compute $\int \frac{1}{9+x^2} dx$.

Solution: Our first priority is to rewrite this so we have a form $1+u^2$ in the denominator.

$$\int \frac{1}{9+x^2} dx = \int \frac{1}{9\left(1+\frac{x^2}{9}\right)} dx = \int \frac{1}{9} \cdot \frac{1}{1+\frac{x^2}{9}} dx.$$

The factor $\frac{1}{9}$ can simply be carried along for the rest of the computation. The denominator of the other factor can be written $1+u^2$ (same as u^2+1 in our formula) if we take $u^2 = \frac{x^2}{9}$, which can be accomplished letting $u = \frac{x}{3}$.

$$\int \frac{1}{9+x^2} dx = \frac{1}{9} \cdot \int \frac{1}{1+\frac{x^2}{9}} \underline{dx} = \frac{1}{9} \int \frac{1}{1+u^2} \cdot \underline{3du}$$

$$\left. \begin{array}{l} u = \frac{x}{3} \\ \Rightarrow du = \frac{1}{3} \underline{dx} \\ \Leftrightarrow 3du = \underline{dx} \end{array} \right\} = \frac{3}{9} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C.$$

The above example illustrated much of the process: algebraically manipulate by factoring to achieve “1” in the appropriate place, and then pick u so the other term is u^2 . It is slightly more complicated with the forms yielding arcsine and arcsecant, due to the presence of the radical. In the next example we will show more detail than one might normally write.

Example 6.7.2 Compute $\int \frac{1}{\sqrt{25-4x^2}} dx$.

Solution: Here we must find a way to replace the constant 25 with 1 instead. We factor as before, but respect the operation of the radical as well.

$$\int \frac{1}{\sqrt{25-4x^2}} dx = \int \frac{1}{\sqrt{25\left(1-\frac{4x^2}{25}\right)}} dx = \int \frac{1}{\sqrt{25}\sqrt{1-\frac{4x^2}{25}}} dx = \int \frac{1}{5} \cdot \frac{1}{\sqrt{1-\frac{4x^2}{25}}} dx.$$

So the factor 25 under the radical becomes the factor 5 outside the radical. Otherwise it is the same process as the previous example. Now we continue, using a substitution which will result in $u^2 = \frac{4x^2}{25}$. For simplicity we take $u = \frac{2x}{5}$.

$$\begin{aligned} \int \frac{1}{\sqrt{25-4x^2}} dx &= \frac{1}{5} \int \frac{1}{\sqrt{1-\frac{4x^2}{25}}} dx = \frac{1}{5} \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{5}{2} du \\ \left. \begin{aligned} u &= \frac{2x}{5} \\ \Rightarrow du &= \frac{2}{5} \cdot dx \\ \Leftrightarrow \frac{5}{2} du &= dx \end{aligned} \right\} &= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} \left(\frac{2x}{5} \right) + C. \end{aligned}$$

This example above again illustrates the role of the number 1 in the denominator, but also suggests a couple of new points that we make here. First, it is not obvious where the factors 2 and 5—being the square roots of the 4 and 25 appearing in the original—will be present in the final answer. There is a pattern for the arctangent form, and a different one for the arcsine form, but patterns can be forgotten if not used often enough, where the logic of manipulating the integral algebraically to get one of the three basic arctrigonometric forms should still be reproducible after the patterns—which we will explore at the end of this section—are forgotten. Second, we are approaching the boundary between integrals which are easily checked with differentiation, and those where the differentiation has at least as many algebraic difficulties as the integration. In such cases, it is usually better to have carefully written each integration step so it can be audited for accuracy, rather than risk algebraic error in testing our answer. Consider a verification of the answer in this latest example (readers’ steps may vary):

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{2} \sin^{-1} \left(\frac{2x}{5} \right) \right] &= \frac{1}{2} \cdot \frac{1}{\sqrt{1-\left(\frac{2x}{5}\right)^2}} \cdot \frac{d}{dx} \left[\frac{2x}{5} \right] = \frac{1}{2} \cdot \frac{1}{\sqrt{1-\frac{4x^2}{25}}} \cdot \frac{2}{5} \\ &= \frac{1}{5} \cdot \frac{1}{\sqrt{1-\frac{4x^2}{25}}} = \frac{1}{\sqrt{25}} \cdot \frac{1}{\sqrt{1-\frac{4x^2}{25}}} = \frac{1}{\sqrt{25-4x^2}}, \quad \text{q.e.d.} \end{aligned}$$

While such a verification is certainly possible, it is not likely one to be performed “mentally” with much confidence, as we may have been able to do with many previous computations. Indeed there are enough constants to be accommodated that this verification should be done in careful writing. In most of Chapter 7 we will see much more complicated rewritings of integrals,

and verification will usually be much better accomplished by checking our individual steps in integration rather than by differentiating of our answers.

Our next example just takes this theme one step further. Recall that substitution in the arcsecant form had a slight complication, which was that the u -variable appeared both inside and outside the radical. This caused a minor complication in Example 6.6.7, page 443 for instance. A similar problem will occur in this next example.

Example 6.7.3 Compute $\int \frac{1}{x\sqrt{81x^2 - 16}} dx$.

Solution: As with the previous two examples, it is necessary to have a 1 in the place presently occupied by 16, so we will factor the 16 from the radical. The other algebraic difficulties will be taken care of by the substitution. Indeed, the remaining term under the radical must be u^2 , and the rest of the form will follow, with residual multiplicative constants.

$$\int \frac{1}{x\sqrt{81x^2 - 16}} dx = \int \frac{1}{4x\sqrt{\frac{81x^2}{16} - 1}} dx = \int \frac{1}{4 \cdot \frac{4u}{9}\sqrt{u^2 - 1}} \cdot \frac{4}{9} du$$

$$\left. \begin{array}{l} u = \frac{9x}{4} \\ \Rightarrow du = \frac{9}{4} \cdot dx \\ \Leftrightarrow \frac{4}{9} du = dx \\ \text{also, } u = \frac{9x}{4} \\ \Leftrightarrow x = \frac{4u}{9} \end{array} \right\} = \int \frac{1}{4u\sqrt{u^2 - 1}} du = \frac{1}{4} \sec^{-1} |u| + C = \frac{1}{4} \sec^{-1} \left| \frac{9x}{4} \right| + C.$$

Note how the term $\frac{81x^2}{16}$ under the radical became simply u^2 , and then the term x outside the radical became $\frac{4u}{9}$, both consistent with $u = \frac{9x}{4}$. Also note that a factor of $\frac{4}{9}$ in the denominator canceled with the same factor multiplying the differential du .

This latest example again illustrates the points made before: that having the 1-term in the denominator is the key to the whole process, that the rest is taken care of by the u -substitution which follows and finally, that checking by differentiation is nontrivial.

Another minor complication is that the numbers we must factor might not be perfect squares. The process is exactly the same, though perhaps some more care is required.

Example 6.7.4 Compute $\int \frac{1}{\sqrt{5 - 2x^2}} dx$.

Solution: The process is exactly the same as before. The key is to factor the denominator to have a 1 in the place of the 5:

$$\int \frac{1}{\sqrt{5 - 2x^2}} dx = \int \frac{1}{\sqrt{5} \cdot \sqrt{1 - \frac{2x^2}{5}}} dx = \frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{1 - u^2}} \cdot \sqrt{\frac{5}{2}} \cdot du$$

$$\left. \begin{array}{l} u = \sqrt{\frac{2}{5}} \cdot x \\ \Rightarrow du = \sqrt{\frac{2}{5}} \cdot dx \\ \Leftrightarrow \sqrt{\frac{5}{2}} \cdot du = dx \end{array} \right\} = \frac{1}{\sqrt{2}} \sin^{-1} u + C = \frac{1}{\sqrt{2}} \sin^{-1} \left(\sqrt{\frac{2}{5}} \cdot x \right) + C.$$

6.7.2 Completing the Square

In the previous subsection our first concern after identifying our target form was to rewrite the integrand to have the number 1 in the appropriate place in the denominator. In this subsection our first concern is identifying, except for a multiplicative constant, what will be u^2 . We do this by completing the square first, and then fixing the form to have the number 1 where we need it, and working from there as before. As there are differing levels of difficulty in such problems, we will begin with one of the simplest and continue from there. It should be noted that the completing the square step is sometimes needed before determining that one of our three forms, (6.85), (6.86) or (6.87), can even be achieved. If not, and we are fortunate, another earlier method may work, though usually we should notice that before attempting the method here. If no earlier method will work, there may be a method available in Chapter 7 that will solve the problem.

Recall that when completing the square, one adds and subtracts $(b/2)^2$, where the original polynomial is $x^2 + bx$, or more generally $x^2 + bx + c$:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c. \end{aligned}$$

As we will see, the fact that the coefficient of x^2 was 1 was key to the computation above. If not, the leading coefficient will be factored from the x^2 and x terms. Our first examples will not require that initial factoring.

Example 6.7.5 Compute $\int \frac{1}{x^2 + 2x + 2} dx$.

Solution: The hope is that we can somehow write the denominator as $1 + u^2$, perhaps multiplied by some nonzero constant, without introducing any more variable factors. For this one we are unusually fortunate. Note that here “ b ” is 2.

$$\begin{aligned} \int \frac{1}{x^2 + 2x + 2} dx &= \int \frac{1}{x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 2} dx = \int \frac{1}{x^2 + 2x + 1 - 1 + 2} dx \\ \left. \begin{aligned} u &= x + 1 \\ \implies du &= dx \end{aligned} \right\} &= \int \frac{1}{(x + 1)^2 + 1} dx = \int \frac{1}{u^2 + 1} du = \tan^{-1} u + C = \tan^{-1}(x + 1) + C. \end{aligned}$$

What made this last example particularly simple was that the additive constant outside of the perfect square was already 1, which is of course key to our arctrigonometric antiderivative forms. If not, we have to perform some division.

Example 6.7.6 Compute $\int \frac{1}{x^2 + 6x + 17} dx$.

Solution: Here $b = 3$, so we add and subtract $\left(\frac{b}{3}\right)^2 = 9$.

$$\begin{aligned} \int \frac{1}{x^2 + 6x + 17} dx &= \int \frac{1}{x^2 + 6x + 9 - 9 + 17} dx = \int \frac{1}{(x + 3)^2 + 8} dx = \frac{1}{8} \int \frac{1}{\frac{(x+3)^2}{8} + 1} dx \\ \left. \begin{aligned} u &= \frac{x + 3}{\sqrt{8}} \\ \implies du &= \frac{1}{\sqrt{8}} dx \\ \iff \sqrt{8} du &= dx \end{aligned} \right\} &= \frac{1}{8} \int \frac{1}{u^2 + 1} \cdot \sqrt{8} du = \frac{1}{\sqrt{8}} \tan^{-1} u + C = \frac{1}{\sqrt{8}} \tan^{-1} \left(\frac{x + 3}{\sqrt{8}} \right) + C. \end{aligned}$$

As this last example illustrates, the final form of the antiderivative can be more complicated when completing the square is required. While it would be an interesting exercise to verify the answer by differentiation, perhaps verifying each individual step in the solution process would be a more efficient means of verifying the answer we derived.

For simplicity we will continue with arctangent forms for the moment, as we look at the next complication, which is that the coefficient of x^2 is not equal to 1. In such a case we factor that leading coefficient out of the entire polynomial, or at least out of the x^2 and x terms. It is then important to perform the addition and subtraction steps of completing the square within the factor with the x^2 and x terms; the addition and subtraction of the $(b/2)^2$ in the process must occur simultaneously and beside each other. Note that such a term has a different effect inside parentheses (or brackets) compared to outside, so we must have the addition and subtraction steps together in order that numerically they have no net effect.

Example 6.7.7 Compute $\int \frac{1}{5x^2 - 4x + 9} dx$.

Solution: Our first priority is to have the coefficient of x^2 to be 1, after which we complete the square and finish the problem.

$$\begin{aligned} \int \frac{1}{5x^2 - 4x + 9} dx &= \int \frac{1}{5 \left[x^2 - \frac{4}{5}x + \frac{9}{5} \right]} dx = \int \frac{1}{5 \left[x^2 - \frac{4}{5} + \left(\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2 + \frac{9}{5} \right]} dx \\ &= \int \frac{1}{5 \left[\left(x - \frac{2}{5}\right)^2 - \frac{4}{25} + \frac{45}{25} \right]} dx = \frac{1}{5} \int \frac{1}{\left[\left(x - \frac{2}{5}\right)^2 + \frac{41}{25} \right]} dx \end{aligned}$$

(Note how we added and subtracted $(2/5)^2$ together, both within the brackets.) Now we need to manipulate this integral so there is a 1 in place of the fraction $\frac{41}{25}$, which we do as before, by factoring. Continuing,

$$\begin{aligned} \int \frac{1}{5x^2 - 4x + 9} dx &= \dots \\ &= \frac{1}{5} \int \frac{1}{\left[\left(x - \frac{2}{5}\right)^2 + \frac{41}{25} \right]} dx \\ &= \frac{1}{5} \int \frac{1}{\frac{41}{25} \left[\underbrace{\frac{25}{41} \left(x - \frac{2}{5}\right)^2}_{u^2} + 1 \right]} dx \\ \Rightarrow \quad \left. \begin{aligned} u &= \frac{5}{\sqrt{41}} \left(x - \frac{2}{5}\right) \\ du &= \frac{5}{\sqrt{41}} dx \\ \Leftrightarrow \frac{\sqrt{41}}{5} du &= dx \end{aligned} \right\} &= \frac{1}{5} \cdot \frac{25}{41} \int \frac{1}{u^2 + 1} \cdot \frac{\sqrt{41}}{5} du \\ &= \frac{1}{\sqrt{41}} \tan^{-1} u + C = \frac{1}{\sqrt{41}} \tan^{-1} \left[\frac{5}{\sqrt{41}} \left(x - \frac{2}{5}\right) \right] + C. \end{aligned}$$

6.8 Hyperbolic Functions

The algebraic and differential structure embedded in the trigonometric functions made for some surprising, but useful derivative and integral formulas involving the arc-trigonometric functions. As it happens, there is another genre of functions with a similar, yet distinct structure, that genre being the so-called *hyperbolic functions*.

In fact the hyperbolic functions are somewhat redundant, in that they are based upon exponential functions and the more interesting integration formulas can be arrived at through other methods. However, exploiting these functions can greatly simplify certain types of integration problems, as we will see.

We will begin with definitions, derivatives and some identities involving the hyperbolic functions. We will then look at their graphs, and consider what their “inverses” would look like, and how they play out in derivative and integral formulas. Along the way we will compare them to their trigonometric counterparts and see how a sign (\pm) here or there can make a crucial difference in an integration problem.

6.8.1 Hyperbolic Functions and Their Basic Intrinsic Structures

We begin with the hyperbolic sine and hyperbolic cosine functions. These can be defined geometrically, but unlike their trigonometric counterparts, these also have straightforward definitions in terms of our earlier, familiar functions (though it is not obvious from the geometry!):

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{2}, \quad (6.88)$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{e^x + e^{-x}}{2}. \quad (6.89)$$

The others are defined in terms of these, just as with the trigonometric functions. Before we define the others, we will notice a couple of relationships which are similar, though distinct, from what occurs with the trigonometric functions. The first result is algebraic:

Theorem 6.8.1 *For all $x \in \mathbb{R}$, we have*

$$\cosh^2 x - \sinh^2 x = 1. \quad (6.90)$$

For the proof, we just expand the left-hand side:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2e^0 + e^{-2x} - e^{2x} + 2e^0 - e^{-2x}}{4} \\ &= \frac{2 + 2}{4} \\ &= 1, \text{ q.e.d.} \end{aligned}$$

Of course this is the hyperbolic analog of the basic trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$. As we will see, the difference in the signs between the trigonometric identity and (6.90) makes for analogous, but significantly distinct results throughout the hyperbolic development. We next look at the derivatives of these:

Theorem 6.8.2 For $x \in \mathbb{R}$, we have

$$\frac{d}{dx} \sinh x = \cosh x, \quad (6.91)$$

$$\frac{d}{dx} \cosh x = \sinh x. \quad (6.92)$$

These are simple chain rule computations. For the first case, we have

$$\frac{d}{dx} \left[\frac{1}{2} (e^x - e^{-x}) \right] = \frac{1}{2} \left(e^x - e^{-x} \frac{d(-x)}{dx} \right) = \frac{1}{2} (e^x - e^{-x}(-1)) = \frac{1}{2} (e^x + e^{-x})$$

i.e., $\frac{d}{dx} \sinh x = \cosh x$. That $\frac{d}{dx} \cosh x = \sinh x$ is similar. Note how these compare to derivative formulas for $\sin x$ and $\cos x$.