Chapter 7

Advanced Integration Techniques

Before introducing the more advanced techniques, we will look at a shortcut for the easier of the substitution-type integrals. Advanced integration techniques then follow: integration by parts, trigonometric integrals, trigonometric substitution, and partial fraction decompositions.

7.1 Substitution-Type Integration by Inspection

In this section we will consider integrals which we would have done earlier by substitution, but which are simple enough that we can guess the approximate form of the antiderivatives, and then insert any factors needed to correct for discrepancies detected by (mentally) computing the derivative of the approximate form and comparing it to the original integrand. Some general forms will be mentioned as formulas, but the *idea* is to be able to compute integrals without resorting to writing the usual u-substitution steps.

Example 7.1.1 *Compute* $\int \cos 5x\,dx$.

Solution: We can anticipate that the **approximate form**[1] *of the answer is* $\sin 5x$*, but then*

$$\frac{d}{dx}\sin 5x = \cos 5x \cdot \frac{d}{dx}(5x) = \cos 5x \cdot 5 = 5\cos 5x.$$

Since we are looking for a function whose derivative is $\cos 5x$*, and we found one whose derivative is* $5\cos 5x$*, we see that our candidate antiderivative* $\sin 5x$ *gives a derivative with an extra factor of* 5*, compared with the desired outcome. Our candidate antiderivative's derivative is 5 times too large, so this candidate* $\sin 5x$ *must be 5 times too large. To compensate and arrive at a function with the proper derivative, we multiply our candidate* $\sin 5x$ *by* $\frac{1}{5}$*. This give us a new candidate antiderivative* $\frac{1}{5}\sin 5x$*, whose derivative is of course* $\frac{1}{5}\cos 5x \cdot 5 = \cos 5x$*, as desired. Thus we have*

$$\int \cos 5x\,dx = \frac{1}{5}\sin 5x + C.$$

It may seem that we wrote more in the example above than with the usual u-substitution method, but what we wrote could be performed mentally without resorting to writing the details.

In future sections, an integral such as the above may occur as a relatively small step in the execution of a more advanced and more complicated method (perhaps for computing a much

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[1] In this section, by *approximate form* we mean a form which is correct except for **multiplicative constants**.

more difficult integral). This section's purpose is to point out how such an integral can be quickly dispatched, to avoid it becoming a needless distraction in the more advanced methods.

Some formulas which should be quickly verifiable by inspection (that is, by reading and mental computation rather than with paper and pencil, for instance) follow:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \tag{7.1}$$

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C, \tag{7.2}$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C, \tag{7.3}$$

$$\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + C, \tag{7.4}$$

$$\int \csc^2 kx\,dx = -\frac{1}{k}\cot kx + C, \tag{7.5}$$

$$\int \sec kx\tan kx\,dx = \frac{1}{k}\sec kx + C, \tag{7.6}$$

$$\int \csc kx\cot kx\,dx = -\frac{1}{k}\csc kx + C, \tag{7.7}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C. \tag{7.8}$$

Example 7.1.2 *The following integrals can be computed with u-substitution, but also are computable by inspection:*

$$\int \frac{1}{5x-9}\,dx = \frac{1}{5}\ln|5x-9| + C,$$
$$\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + C,$$
$$\int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C,$$
$$\int \sec^2 \pi x\,dx = \frac{1}{\pi}\tan\pi x + C,$$
$$\int \csc 6x\cot 6x\,dx = -\frac{1}{6}\csc 6x + C.$$

While it is true that we can call upon the formulas (7.1)–(7.8), the more flexible strategy is to anticipate the form of the antiderivative and adjust accordingly. For instance, we have the following antiderivative form, written two ways:

$$\int \frac{1}{u}\,du = \ln|u| + C,$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

(As usual, the second form is the same as the first where $u = f(x)$.) So when we see an integrand which is a fraction, the numerator being the derivative of the denominator except for multiplicative constants, we know the antiderivative will be, approximately, the natural log of the absolute value of that denominator.

Example 7.1.3 *Consider* $\int \frac{x}{x^2+1}\,dx$

Here we see that the derivative of the denominator is also a factor in the integrand. Our candidate approximate form can then be $\ln|x^2+1| = \ln(x^2+1)$*. Now we differentiate to see what we need to include to get the correct derivative:*

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x = 2\cdot\frac{x}{x^2+1}.$$

To correct for the extra factor of 2 and thus get the correct derivative, we insert the factor $\frac{1}{2}$*:*

$$\frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right] = \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x = \frac{x}{x^2+1},$$

as desired. Thus

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+C.$$

To be sure, a quick (mental?) check by differentiation verifies the answer.

Of course there are many other forms.

Example 7.1.4 *Consider* $\int \frac{1}{\sqrt{5x-9}}\,dx.$

Of course this can be rewritten $\int(5x-9)^{-1/2}\,dx$*. Now it is crucial that a complete substitution,* $u = 5x-9 \implies du = 5\,dx$*, etc., shows that* du *and* dx *agree except for a multiplicative constant, so we know that the integral—up to multiplicative constants—is of form* $\int u^{-1/2}\,du$*, which is a power rule.*

The approximate *form of the antiderivative is thus* $u^{1/2} = (5x-9)^{1/2}$*, which we write in* x *and then differentiate,*

$$\frac{d}{dx}(5x-9)^{1/2} = \frac{1}{2}(5x-9)^{-1/2}\cdot 5,$$

which has **extra** *factors (compared to our original integrand) of collectively* $\frac{5}{2}$*. To cancel their effects we include a factor* $\frac{2}{5}$ *in our actual, reported antiderivative. Thus*

$$\int \frac{1}{\sqrt{5x-9}}\,dx = \frac{2}{5}(5x-9)^{1/2}+C = \frac{2}{5}\sqrt{5x-9}+C.$$

Note that a quick derivative computation, albeit involving a (simple) chain rule, gives us the correct function $1/\sqrt{5x-9}$.

Example 7.1.5 *Consider* $\int 7x\,\sin^5 x^2\cos x^2\,dx.$

For such an antiderivative, our ability to guess the form depends upon our exptertise with the original substitution methods. In all of these was a form $\int f(u)\,K\,du$*, where we could anticipate both* u *and* f*, with* du *accounting for remaining terms, and* $K\in\mathbb{R}$ *which we can ignore by taking our shortcut path described in this section. Looking ahead, the student well-versed in substitution will expect* $u = \sin x^2$*, and the integral being of the approximate form* $\int u^5\,du$*. Thus we will have an approximate antiderivative of* u^6 *(times a constant), i.e., the approximate form should be* $\sin^6 x^2$*. Now we differentiate this and see what compensating factors must be included to reconcile with the original integrand:*[2]

$$\frac{d}{dx}(\sin x^2)^6 = 6(\sin x^2)^5\cdot\cos x^2\cdot 2x = 12x\,\sin^5 x^2\cos x^2.$$

Of course we want 7 *in the place of the* 12 *(or separately,* $2 \cdot 6$*), so we multiply by* $\frac{7}{12}$ *(or again,* $7 \cdot \frac{1}{6 \cdot 2}$*). With this we have*

$$\int 7x \sin^5 x^2 \cos x^2 \, dx = \frac{7}{12} \sin^6 x^2 + C.$$

It would be perfectly natural to forego this method of "guess and adjust" in favor of the old-fashioned substitution method. Indeed the full substitution method has some advantages (see the next subsection). For instance, it is more "constructive," and thus less error-prone; one is less tempted to skip steps while employing substitution, while one might attempt a mental derivation of the answer here and thus easily be off by a factor. It is important that each student find the comfortable level of brevity for himself.[3]

The method used in the above examples can be summarized as follows:

1. Anticipate the form of the antiderivative by an *approximate form* (correct up to a multiplicative constant).
2. Differentiate this approximate form and compare to the original function (to be integrated);
3. If Step 1 is correct, and thus the approximate form's derivative differs from the original (integrand) function by a multiplicative constant, insert a compensating, reciprocal multiplicative constant in the approximate form to arrive at the actual antiderivative;
4. For verification, differentiate the answer to see if the original function emerges.

Example 7.1.6 *Compute* $\displaystyle\int x^3 \sin x^4 \, dx$.

Solution: This is of the approximate form $\int \sin u \, du$*, with* $u = x^4$*. The approximate form of the solution is thus* $\cos x^4 + C$ *(or* $-\cos x^4 + C$*, but these differ by a multiplicative constant* -1*), which has derivative* $-\sin x^4 \cdot 4x^3$*. We introduce a factor of* $-\frac{1}{4}$ *to compensate for the extra factor of* -4*:*

$$\int x^3 \sin x^4 \, dx = -\frac{1}{4} \cos x^4 + C,$$

which can be quickly verified by differentiation.

Example 7.1.7 *Compute* $\displaystyle\int x\sqrt{9 - x^2} \, dx$.

Solution: It is advantageous to read this integral $\int x(9 - x^2)^{1/2} \, dx$*, which is of approximate form* $\int u^{1/2} \, du$ *(where* $u = 9 - x^2$*). These observations, and the approximate form* $(9 - x^2)^{3/2}$ *of the integral, can be gotten by mental observation (referred to earlier as "by inspection"). Its derivative is* $\frac{3}{2}(9 - x^2)^{1/2} \cdot (-2x)$*, which has an extra factor of* -3 *(after cancellation). Thus*

$$\int x\sqrt{9 - x^2} = -\frac{1}{3} \left(9 - x^2\right)^{3/2} + C.$$

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[2] Notice that we are assuming fluency in the chain rule as we compute the derivative of $\sin^6 x^2$, rather than writing out every step as we did in Chapter 4. Each student must gage personal ability to omit steps.

[3] It is the author's experience that students in engineering and physics programs are more interested in arriving at the answer quickly, while mathematics and other science students prefer the presentation of the full substitution method. The latter are somewhat less likely to be wrong by a multiplicative constant, though the former tend to progress through the topics faster. There are, of course, spectacular exceptions, and each group benefits from camaraderie with the other.

7.1.1 Limitations of the Method

There are two very important points to be made about the limitiations of the method. The first point is argued by making several related points, and the second is illustrated in an example.

(I) This method can not totally replace the earlier substitution method.

- (a) The skills used in the substitution method will be needed for later methods. In particular, the idea that the entire integral in x is replaced by one in u (for instance), including the dx and, if a definite integral, the interval of integration.
- (b) If an integral is difficult enough, the more constructive substitution method is less error-prone than this "guess and adjust" style here.
- (c) The idea of the substitution method is the same as this method; anticipating what to set equal to u is equivalent to guessing the approximate form of the integral in u, and thus the approximate form of the antiderivative.
- (d) When using numerical and other methods with definite integrals, a substitution can sometimes make for a much simpler integral to be approximated or otherwise analyzed, even if the antiderivative is never computed. For instance, with $u = x^2$, giving then $du = 2x\,dx$, we can write

$$\int_{-1}^{2} xe^{x^4}\,dx = \frac{1}{2}\int_{1}^{4} e^{u^2}\,du.$$

(II) It is imperative that the derivative of the approximate form differs from the original function to be integrated by at most a multiplicative constant. In particular, an extra variable function cannot be compensated for.

To illustrate this point, and simultaneously warn against a common mistake, consider

$$\int \frac{1}{x^2+1}\,dx.$$

The mistake to avoid here is to take the approximate solution to be $\ln(x^2+1)$, which we then notice has derivative

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x.$$

Unfortunately we cannot compensate by dividing by the extra factor $2x$, because[4]

$$\frac{d}{dx}\left[\frac{\ln(x^2+1)}{2x}\right] = \frac{2x\cdot\frac{d\ln(x^2+1)}{dx} - \ln(x^2+1)\cdot\frac{d(2x)}{dx}}{(2x)^2},$$

which is guaranteed (by the presence of the logarithm in the result) to be something other than our original function $\frac{1}{x^2+1}$. The method does not work because multiplicative *functions* do not "go along for the ride" in derivative (or antiderivative) problems the way multiplicative constants do.

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[4] Alternatively, a product rule computation can be used:

$$\frac{d}{dx}\left[\frac{1}{2x}\ln(x^2+1)\right] = \frac{1}{2x}\cdot\frac{d\ln(x^2+1)}{dx} + \ln(x^2+1)\cdot\frac{d}{dx}\left[\frac{1}{2x}\right],$$

which eventually gives the original function for the first product, but the second part of the product rule is a complication we cannot rid ourselves of easily.

It should be pointed out that the method of the next section does utilize the fact that the first product above is the desired original function, and an algorithm can be fashioned to compensate for the presence of the second product. The application of that method is not universally useful, and even when it is helpful it takes considerable work to develop the theory as well as fluency in its application.

Of course we knew from before that

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C,$$

so this integral is not really suitable for a substitution argument, but is rather a special case in and of itself.

Exercises

1. $\displaystyle\int (x^2+1)^7 \cdot 2x\,dx$
2. $\displaystyle\int \cos x^4 \cdot 4x^3\,dx$
3. $\displaystyle\int 15x^2 \sec^2 5x^3\,dx$
4. $\displaystyle\int \frac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}\,dx$
5. $\displaystyle\int \frac{\csc^2\left(\frac{1}{x}\right)}{x^2}\,dx$
6. $\displaystyle\int \tan^7 x\,\sec^2 x\,dx$
7. $\displaystyle\int \frac{x}{(x^2+1)^3}\,dx$
8. $\displaystyle\int (2x-11)^9\,dx$
9. $\displaystyle\int \cos 5x\,dx$
10. $\displaystyle\int \sec 9x \tan 9x\,dx$
11. $\displaystyle\int \cos x \sin x\,dx$ (See #13)
12. $\displaystyle\int \tan^3 5x \sec^2 5x\,dx$
13. $\displaystyle\int \sin x \cos x\,dx$ (See #11)
14. $\displaystyle\int \sin^3 x \cos x\,dx$
15. $\displaystyle\int \tan^5 x \sec^2 x\,dx$
16. $\displaystyle\int x \sin x^2\,dx$
17. $\displaystyle\int x^3 \cdot \sqrt{x^4-2}\,dx$
18. $\displaystyle\int (x^3+x^2)^4(3x^2+2x)\,dx$
19. $\displaystyle\int \sec^5 x \cdot \sec x \tan x\,dx$
20. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
21. $\displaystyle\int x^2 \sin x^3 \cos^5 x^3\,dx$
22. $\displaystyle\int \frac{\sin^3\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}{x^2}\,dx$
23. $\displaystyle\int e^x \cos e^x\,dx$
24. $\displaystyle\int x e^{x^2}\,dx$
25. $\displaystyle\int e^{2x} \sin e^{2x}\,dx$
26. $\displaystyle\int e^{-x} \sec^2 e^{-x}\,dx$
27. $\displaystyle\int e^{5x}\,dx$
28. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$
29. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$
30. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$ (Hint: multiply integrand by e^x/e^x.)
31. $\displaystyle\int e^{4x}(9+e^{4x})^{10}\,dx$
32. $\displaystyle\int x e^{-2x^2}\,dx$

33. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

34. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

35. $\displaystyle\int e^{3\cos 2x}\sin 2x\,dx$

36. $\displaystyle\int \frac{\cos x}{\sin x + 1}\,dx$

37. $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

38. $\displaystyle\int \frac{\sin x}{\cos x}\,dx$

39. $\displaystyle\int \frac{2x+1}{x^2+x}\,dx$

40. $\displaystyle\int \frac{x}{x^2+1}\,dx$

41. $\displaystyle\int \frac{1}{x^2+1}\,dx$

42. $\displaystyle\int \frac{1}{x\ln x}\,dx$

43. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

44. $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\,dx$

46. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

47. $\displaystyle\int \frac{\ln x}{x}\,dx$

48. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

49. $\displaystyle\int \frac{1}{x(1+(\ln x)^2)}\,dx$

50. $\displaystyle\int \frac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}\,dx$

51. $\displaystyle\int \frac{\sec^2(\ln x)}{x}\,dx$

52. $\displaystyle\int \frac{(9+\ln x)^6}{x}\,dx$

53. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

7.2 Integration By Parts

While integration by substitution in its elementary form takes advantage of the chain rule, by contrast integration by parts exploit the product rule. The application is a bit more complicated than with substitution, and there are perhaps more variations on the theme than with substitution. Furthermore, to be truly fluent in this method requires one to be able to see more steps ahead than with substitution, as the method is arguably twice as long and complicated as most substitution problems. Still, it can be similarly mastered with practice.

7.2.1 The Idea by an Example

Suppose that we need to find an antiderivative of the function $f(x) = x\sec^2 x$. It is not hard to see that normal substitution in not going to easily yield our desired antiderivative F:

$$\int x\sec^2 x\,dx = F(x) + C.$$

However, a clever student might notice that $x\sec^2 x$ contains terms that could have arisen from a product rule derivative computation:

$$\begin{aligned}\frac{d}{dx}\left[x\tan x\right] &= x\cdot\frac{d\tan x}{dx} + \tan x\cdot\frac{dx}{dx}\\ &= x\sec^2 x + \tan x.\end{aligned}$$

If we rearrange the terms above, we can summarize as follows:

$$x\sec^2 x = \frac{d}{dx}\left[x\tan x\right] - \tan x.$$

In fact the line immediately above is perhaps where the spirit of the method is most on display: that the given function is indeed one *part* of a product rule derivative. If we are fortunate, the other *part* of the product rule formula is easier to integrate, because the derivative term, namely $\frac{d}{dx}\left[x\tan x\right]$ is trivial to integrate, since we are just asking for the antiderivative of the derivative of a (differentiable) function, which ultimately returns the function itself, and an additive constant we can absorb in the second integral. Indeed, if we take antiderivatives of both sides, we then get

$$\begin{aligned}\int x\sec^2 x\,dx &= \int\left[\left(\frac{d[x\tan x]}{dx}\right) - \tan x\right]\,dx\\ &= x\tan x - \int\tan x\,dx\\ &= x\tan x - \ln|\sec x| + C.\end{aligned}$$

From such as the above emerges a method whereby we identify our given function (here $x\sec^2 x$) as a *part* of a product rule computation ($\frac{d}{dx}[x\tan x]$), and integrate our original function by instead (trivially) integrating the product rule derivative term (again $\frac{d}{dx}[x\tan x]$), and then integrating the other *part* ($\tan x$) of the product rule output. Often the other, hidden part of the underlying product rule is easier to integrate than the original function, and therein lies much of the success of the method.

7.2.2 The Technique In Its Simpler Applications

Recall that when we completely developed the substitution method, the underlying principle—the chain rule—was not written out in complete derivative form, but rather in *differential* form. That was partly because of the compactness of the differential notation. Having supposed that F was an antiderivative of f, we eventually settled on writing the argument below without the first two integrals:

$$\int f(u(x))u'(x)\,dx = \int f(u(x))\cdot\frac{du(x)}{dx}\,dx = \int f(u)\,du = F(u)+C = F(u(x))+C.$$

At first we did write the first steps because the proof was in the chain rule: $\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$. However, we eventually opted for the differential form, though for most it takes some practice. We will adopt differential notation in integration by parts as well. For instance, recall the product rule can be rewritten in differential form:

$$d(uv) = u\,dv + v\,du. \tag{7.9}$$

Of course this came from multiplying a derivative product rule by, say, dx, assuming u and v are in fact functions of x:

$$\frac{d(uv)}{dx} = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}.$$

Now we rearrange (7.9) as follows:

$$u\,dv = d(uv) - v\,du. \tag{7.10}$$

Equation (7.10) is perhaps the best equation to visualize the principle behind the eventual integration formula, because it is an easy step from the product rule. The actual formula quoted in most textbooks is still two steps away. First we integrate both sides:

$$\int u\,dv = \int d(uv) - \int v\,du. \tag{7.11}$$

Next we notice that the first integral on the right hand side of (7.11) is simply uv, and so we arrive at our final working formula for our integration by parts technique:

$$\boxed{\int u\,dv = uv - \int v\,du.} \tag{7.12}$$

Most textbooks and instructors use the formula above in exactly that form. It is best memorized, though its derivation—particuarly from (7.10)—should not be forgotten.

Next we look at an example of the actual application of (7.12). In the example below, the arrangement of terms is as one would work the problem with pencil and paper, except for the implication arrows and the underbraces (which we explain briefly below, and exclude from then on).

Example 7.2.1 *Compute* $\int xe^x\,dx.$

Solution:

$$\begin{array}{cc} u = x & dv = e^x\,dx \\ \Downarrow & \Uparrow \\ du = dx & v = e^x \end{array}$$

$$\int \underbrace{x}_{u}\ \underbrace{e^x\,dx}_{dv} = uv - \int v\,du = (x)(e^x) - \int (e^x)\,dx = xe^x - e^x + C.$$

It is interesting to note that we *choose* u and dv, and then *compute* du and v, with one qualification. That is that v is not unique; the computation from dv to v is of an antidifferentiation nature and so we really only know v up to an additive constant. In fact *any* v so that $dv = e^x\,dx$ (we took $v = e^x \implies dv = e^x\,dx$) will work in (7.12). Any additive constant, while legitimate, will eventually cancel in the final computation. For instance, if we had chosen $v = e^x + 100$, we would have had

$$\begin{aligned} uv - \int v\,du &= x(e^x + 100) - \int (e^x + 100)\,dx \\ &= xe^x + 100x - e^x - 100x + C \\ &= xe^x - e^x + C, \end{aligned}$$

as before. For most cases, we will just assume that the additive constant is zero and we will use the simplest antiderivative for v. (We will not continue to write the implication arrows as they are technical and perhaps confusing.)

Now we will revisit the example we gave in Subsection 7.2.1, using what will be our basic style for this method.

Example 7.2.2 *Compute* $\displaystyle\int x\sec^2 x\,dx$.

Solution:

$$\begin{aligned} u &= x & dv &= \sec^2 x\,dx \\ du &= dx & v &= \tan x \end{aligned}$$

$$\int x\sec^2 x\,dx = \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x| + C.$$

Since this method is more complicated than substitution, there are more complicated considerations in how to apply it. First of course, one should attempt an earlier, simpler method. But if those fail, and integration by parts is to be attempted,[5] the following guidelines for choosing u and dv should be considered:

1. u and dv **must** account for exactly all factors of the integral.

    1.5. Of course, dv **must** contain the differential term (for example, dx) as a factor, but can contain more terms.

2. $v = \int dv$ should be computable with relative ease.
3. $du = u'(x)\,dx$ (assuming the original integral was in x) should not be overly complicated.
4. The integral $\int v\,du$ should be simpler than the original integral $\int u\,dv$.[6]

The next example illustrates the importance of the second consideration (numbered 2.) above.

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[5] Of course with practice one can see ahead whether or not integration by parts is likely to achieve an answer for a particular integral.

[6] Later, in a twist on the method, we will see that the we do not necessarily require $\int v\,du$ be easier than the original, $\int u\,dv$.

Example 7.2.3 *Compute* $\displaystyle\int x^3\sin x^2\,dx$.

Solution: We do not want to make $u=\sin x^2$, because then $dv=x^3\,dx$, giving $du=2x\cos x^2$ and $v=\frac{1}{4}x^4$, and our $\int v\,du$ will be $\int\frac{1}{2}x^5\cos x^2\,dx$, which is worse than our original integral.

We will instead take u to be some power of x, but not all of x^3, else the terms remaining for dv would be $dv=\sin x^2\,dx$, which we cannot integrate with ordinary methods.[7]

What we will settle on is $dv=x\sin x^2\,dx$, for its integral is an easy substitution. We leave the remaining terms, x^2, for u:

$$\begin{aligned} u &= x^2 & dv &= x\sin x^2\,dx \\ du &= 2x\,dx & v &= -\frac{1}{2}\cos x^2 \end{aligned}$$

$$\begin{aligned} \int x^3\sin x^2\,dx &= \int \underbrace{x^2}_{u}\cdot\underbrace{x\sin x^2\,dx}_{dv} \\ &= uv-\int v\,du \\ &= (x^2)\left(-\frac{1}{2}\cos x^2\right)-\int\left(-\frac{1}{2}\cos x^2\right)2x\,dx \\ &= -\frac{x^2}{2}\cos x^2+\frac{1}{2}\int\cos x^2\cdot x\,dx \\ &= -\frac{x^2}{2}\cos x^2+\frac{1}{2}\sin x^2+C. \end{aligned}$$

We omitted the details of computing v from dv, and computing the last integral, as most students at this point can anticipate the approximate forms of those antiderivatives, mentally compute the derivatives of the approximate forms, and then insert constant factors required to make the antiderivatives correct. (This was the purpose of the last section.)

This last example shows how the requirement that $v=\int dv$ (up to an additive constant) be computable helps to guide us to the proper choice of u and dv. It was lucky that the second integral was easily computable (which would not have been the case if the original integral were, say, $\int x^2\sin x^2\,dx$ or $\int x^4\sin x^2\,dx$), but anyhow we cannot even get to the second integral if we cannot compute v.

The next example shows a different lesson: that it is sometimes appropriate to integrate by parts more than once in a given problem.

Example 7.2.4 *Compute* $\displaystyle\int x^2\cos 3x\,dx$.

Solution: For reasons that will be clear later, we will call this integral $(\mathcal{I})$. *Now we proceed to the integration by parts.*

$$\begin{aligned} u &= x^2 & dv &= \cos 3x\,dx \\ du &= 2x\,dx & v &= \frac{1}{3}\sin 3x \end{aligned}$$

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[7] In fact, we cannot compute $\int\sin x^2\,dx$ using any kind of substitution or parts, or any other method of this text for that matter, and arrive at an antiderivative in simple terms of the functions we know so far such as powers, exponentials, logarithms, trigonometric or hyperbolic functions or their inverses. However, when we study series we will find other expressions with which we can fashion an antiderivative of $\sin x^2$.

$$(\mathcal{I}) = \int \underbrace{x^2}_{u} \underbrace{\cos 3x\,dx}_{dv} = uv - \int v\,du = \frac{1}{3}x^2 \sin 3x - \int \frac{2}{3} x \sin 3x\,dx.$$

While we still cannot compute this last integral directly with old methods, it is better than the original in the sense that our trigonometric function is multiplied by a first-degree polynomial, where in the original the polynomial was second degree. The work which is left more closely resembles our earliest examples of integration by parts.

A strict use of the language would force us to introduce two new variables other than u *and* v*, but since they have "disappeared" in the present form of our answer, namely* $\frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\,dx$*, it is not considered such bad form to "reset" (or "recycle")* u *and* v *for another integration by parts step, this time involving the integral* $\int x \sin 3x\,dx$*:*

$$\begin{aligned} u &= x & dv &= \sin 3x\,dx \\ du &= dx & v &= -\frac{1}{3}\cos 3x \end{aligned}$$

$$\int \underbrace{x}_{u} \underbrace{\sin 3x\,dx}_{dv} = uv - \int v\,du = -\frac{x}{3}\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{x}{3}\cos 3x + \frac{1}{3}\cdot\frac{1}{3}\sin 3x + C_1.$$

Now we insert this last result into our original computation:

$$\begin{aligned} \int x^2 \cos 3x\,dx &= \frac{x^2}{3}\sin 3x - \frac{2}{3}\left[-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C_1\right] \\ &= \frac{x^2}{3}\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C. \end{aligned}$$

Several lessons can be gleaned from the example above. (1) it is very important for proper "bookkeeping," as these problems can beget several "subproblems," and the proper placement of resulting terms is crucial to getting the correct answer; (2) it is not unknown to use integration by parts more than once in a problem; (3) if we can take as many antiderivatives of a function $f(x)$ as we like (i.e., the antiderivative, the antiderivative of the antiderivative, etc.), then for an integral $\int x^n f(x)\,dx$ we can let $u = x^n$, and integration by parts will yield a second integral with a reduction in the power of x, namely

$$\int x^n f(x)\,dx = x^n F(x) - \int n x^{n-1} F(x)\,dx \tag{7.13}$$

where $F' = f$.

Other cases concern choices of u where we choose a function whose derivative we know, but whose antiderivative might not be standard knowledge for the average student.

Example 7.2.5 *Compute* $\displaystyle\int (x^2+1)\ln x\,dx.$

Solution: We cannot let $dv = \ln x\,dx$*, since as yet we do not know the antiderivative of* $\ln x$*. (Even if we did, such a choice for* dv *would not be advantageous, as Example 7.2.6 helps to show.) So we have little choice but to let* $\ln x$ *be* u*.*

$$\begin{aligned} u &= \ln x & dv &= (x^2+1)\,dx \\ du &= \frac{1}{x}\,dx & v &= \frac{x^3}{3} + x \end{aligned}$$

$$\begin{aligned}\int (x^2+1)\ln x\,dx &= uv-\int v\,du\\ &= (\ln x)\left(\frac{x^3}{3}+x\right)-\int\left(\frac{x^3}{3}+x\right)\frac{1}{x}\,dx\\ &= \frac{1}{3}(\ln x)(x^3+3x)-\int\left(\frac{x^2}{3}+1\right)\,dx\\ &= \frac{1}{3}(x^3+3x)\ln x-\frac{1}{9}x^3-\frac{1}{3}x+C.\end{aligned}$$

7.2.3 A Simple Twist on the Method

Here we show how to find antiderivatives of interesting functions whose derivatives we already know.

Example 7.2.6 *Compute* $\int \ln x\,dx$.

Solution: Here we cannot let $dv=\ln x\,dx$*, for computing* $v=\int dv$ *would be the same as computing the whole, original integral. As in Example 7.2.5, we also note that placing* $\ln x$ *in the* u*-term makes for a simpler derivative. Thus we write*

$$\begin{aligned} u&=\ln x & dv&=dx\\ du&=\frac{1}{x}\,dx & v&=x,\end{aligned}$$

so that

$$\begin{aligned}\int \ln x\,dx &= uv-\int v\,du\\ &= (\ln x)(x)-\int (x)\left(\frac{1}{x}\,dx\right)\\ &= x\ln x-\int 1\,dx\\ &= x\ln x-x+C.\end{aligned}$$

In fact the same type of computation will be used for finding antiderivatives of arctrigonometric functions.

Example 7.2.7 *Compute* $\int \sin^{-1}x\,dx$.

Solution: Again we have no choice but to let $u=\sin^{-1}x$*,* $dv=dx$*.*

$$\begin{aligned} u&=\sin^{-1}x & dv&=dx\\ du&=\frac{1}{\sqrt{1-x^2}}\,dx & v&=x\end{aligned}$$

For brevity, we will begin to label the desired integral $(\boldsymbol{\mathcal{I}})$*, so here* $(\boldsymbol{\mathcal{I}})=\int\sin^{-1}x\,dx$*. (The second interval is computed "by inspection.")*

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) &= uv-\int v\,du = x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx\\ &= x\sin^{-1}x+\left(1-x^2\right)^{1/2}+C.\end{aligned}$$

7.2.4 An Indirect Method

The following method, which we describe in the end, is useful in surprisingly many settings.

Example 7.2.8 *Compute* $\displaystyle\int e^{2x}\cos 3x\,dx = (\mathcal{I})$.

<u>Solution</u>: Here we again name the desired integral $(\mathcal{I})$ *for brevity in later steps.*

It should be obvious (especially after a few attempts) that simple substitution methods will not work. So we attempt an integration by parts.

Step 1. *We will let the trigonometric function be part of* dv*:*

$$\begin{aligned} u &= e^{2x} & dv &= \cos 3x\,dx \\ du &= 2e^{2x}\,dx & v &= \frac{1}{3}\sin 3x \end{aligned}$$

So far, after some rearrangement and simplifying, we have

$$\begin{aligned} (\mathcal{I}) &= uv - \int v\,du \\ &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\underbrace{\int e^{2x}\sin 3x\,dx}_{(\mathcal{II})}\,. \end{aligned} \tag{7.14}$$

This does not seem any easier than the first integral, so perhaps we might continue, but this time let the trigonometric function be u *and the exponential (along with* dx*) be contained in* dv*.*

Step 2. *Compute* $(\mathcal{II}) = \int e^{2x}\sin 3x\,dx$ *in light of the comments at the end of the first step.*

$$\begin{aligned} u &= \sin 3x & dv &= e^{2x}\,dx \\ du &= 3\cos 3x\,dx & v &= \frac{1}{2}e^{2x} \end{aligned}$$

$$(\mathcal{II}) = uv - \int v\,du = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx.$$

Combining this with the conclusion (7.14) of Step 1 gives us:

$$\begin{aligned} (\mathcal{I}) &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right] \\ &= \frac{1}{3}e^{2x}\sin 3x - \frac{1}{3}e^{2x}\sin 3x + \int e^{2x}\cos 3x\,dx \\ &= \int e^{2x}\cos 3x\,dx. \end{aligned}$$

Unfortunately that puts us right back where we started. However, a minor change in our effort above will eventually lead us to the solution. While keeping Step 1, our next step towards a solution is to replace Step 2 by the same strategy as used in Step 1, namely that we use the exponential function for u *and the trigonometric with* dv*.*

Step 2—Second Attempt *Again we compute* $(\mathcal{II}) = \int e^{2x}\sin 3x\,dx$*:*

$$\begin{array}{ll} u = e^{2x} & dv = \sin 3x\,dx \\ du = 2e^{2x}\,dx & v = -\dfrac{1}{3}\cos 3x \end{array}$$

$$\begin{aligned}(\mathcal{II}) &= uv - \int v\,du \\ &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx. \qquad (7.15)\end{aligned}$$

It may seem that (7.15) is also a dead end, since it contains the original integral. But this attempt is different. In fact, when we combine (7.15) with (7.14) we get

$$\begin{aligned}(\mathcal{I}) &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}(\mathcal{II}) \\ &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx\right] \\ &= \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\underbrace{\int e^{2x}\cos 3x\,dx}_{(\mathcal{I})},\end{aligned}$$

which we can summarize by the following equation:

$$(\mathcal{I}) = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}(\mathcal{I}). \qquad (7.16)$$

Now we are ready to derive $(\mathcal{I})$*, not by another calculus computation, but in fact by simple algebra.*

Step 3. *Solve (7.16) for* $(\mathcal{I})$*. First we add* $\frac{4}{9}(\mathcal{I})$ *to both sides.*

$$\frac{13}{9}(\mathcal{I}) = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + C_1.$$

Here we add C_1 *because in fact each* $(\mathcal{I})$ *in (7.16) represents all antiderivatives which of course differ from eachother by additive constants. Now (7.16) makes sense because of the fact that there are additive constants on both sides of that equation (though on the right side they are mulitplied by* $-\frac{4}{9}$*, but that still yields additive constants).*[8] *Solving for* $(\mathcal{I})$ *we now have*

$$\begin{aligned}(\mathcal{I}) &= \frac{9}{13}\left[\frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + C_1\right] \\ &= \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C, \qquad (7.17)\end{aligned}$$

where $C = \frac{9}{13}C_1$.

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[8] In fact, two simlutaneous appearances of $(\mathcal{I})$ do not have to have the same additive constants, so $(\mathcal{I}) - (\mathcal{I}) = C_2$, not zero.

What is important to understand about the example above is that sometimes, though we cannot perhaps directly compute a particular integral, it may happen that an indirect method gives us the answer. Here we found an equation, namely (7.16), which our desired integral satisfies, and for which it could be solved algebraically. We must be open to the possibility of finding a desired quantity by indirect methods, as well as direct computations.

It should be pointed out that we could have computed the integral in Example 7.2.8 by instead letting u be the trigonometric function, and $dv = e^{2x}\,dx$ in *both* steps. In fact it is usually best to pick similar choices for u and dv when an integration by parts will take more than one step. (Recall the discussion for $\int x^n f(x)\,dx$.)

The method of Example 7.2.8, namely solving for $(\boldsymbol{\mathcal{I}})$ after an integration by parts step, is available perhaps more often than one would think.

Example 7.2.9 *Compute* $\displaystyle\int \sin^2 x\,dx = (\boldsymbol{\mathcal{I}})$.

Solution:

$$\begin{aligned} u &= \sin x & \qquad dv &= \sin x\,dx \\ du &= \cos x\,dx & v &= -\cos x \end{aligned}$$

$$(\boldsymbol{\mathcal{I}}) = \int uv - \int v\,du = -\sin x\cos x + \int \cos^2 x\,dx.$$

We could perform the same integration by parts with the second integral, but instead we will use the fact that $\cos^2 x = 1 - \sin^2 x$*:*

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= -\sin x\cos x + \int (1-\sin^2 x)\,dx \\ &= -\sin x\cos x + x - \int \sin^2 x\,dx \\ &= x - \sin x\cos x - (\boldsymbol{\mathcal{I}}). \end{aligned}$$

Here we see that we can add $(\boldsymbol{\mathcal{I}})$ *to both sides, divide by* 2 *and get*

$$(\boldsymbol{\mathcal{I}}) = \frac{1}{2}(x - \sin x\cos x) + C.^{9}$$

The example above can be computed directly if we use the trigonometric fact that $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, giving

$$\begin{aligned} \int \sin^2 x\,dx &= \int \frac{1}{2}(1-\cos 2x)\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C \\ &= \frac{1}{2}x - \frac{1}{4}\cdot 2\sin x\cos x + C, \end{aligned}$$

which is the same as before. For the last step we used the (double-angle) trigonometric identity $\sin 2x = 2\sin x\cos x$. In Section 7.3 we will opt for this alternative method, and indeed will make quite an effort to exploit the algebraic properties of the trignometric functions wherever possible, but some integrals there will also *require* integration by parts.

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[9] The intermediate step would be $2(\boldsymbol{\mathcal{I}}) = x - \sin x\cos x + C_1$. See the discussion for Example 7.2.8, page 714. In fact many textbooks do not bother writing the C_1 term, preferring to remind the student at the end that an indefinite integral problem necessitates a "$+C$."

7.2.5 Miscellaneous Considerations

First we look at a definite integral arising from integration by parts. It should be pointed out that the general formula will look like the following:[10]

$$\int_{x=a}^{x=b} u\,dv = uv\bigg|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du. \tag{7.18}$$

Example 7.2.10 *Compute* $\displaystyle\int_{-\pi}^{\pi} x\sin x\,dx$.

<u>Solution</u>: *(Note how the negative signs cancel in the parts formula step.)*

$$\begin{aligned} u &= x & dv &= \sin x\,dx\\ du &= dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} x\sin x\,dx &= (-x\cos x)\big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x\,dx\\ &= [-\pi\cos\pi] - [-(-\pi)\cos(-\pi)] + \sin x\bigg|_{-\pi}^{\pi}\\ &= (-\pi)(-1) - (\pi)(-1) + \sin\pi - \sin(-\pi)\\ &= \pi + \pi + 0 - 0\\ &= 2\pi.\end{aligned}$$

In the example above, We could also have noticed that $\int_{-\pi}^{\pi}\cos x\,dx$ is zero because we are integrating over a whole period $[-\pi,\pi]$ of $\cos x$, and both $\sin x$ and $\cos x$ have definite integral zero over any full period.

It is typical to compute that part $u(x)v(x)\big|_{z}^{b}$ separately, but one could instead and separately compute the entire antiderivative, and then evaluate at the two limits and take the difference:

$$\int_{-\pi}^{\pi} x\sin x\,dx = (-x\cos x + \sin x)\bigg|_{-\pi}^{\pi}.$$

Which method to use is a matter of bookkeeping preferences, and perhaps whether or not part of the right-hand side of (7.18) is particularly simple.

The next example gives us several options along the way, though in each the original choices of u and dv are the same.

Example 7.2.11 *Compute* $\displaystyle\int x\tan^{-1}x\,dx = (\boldsymbol{\mathcal{I}})$.

<u>Solution</u>: *Again we have little choice on our selection of u and dv.*

$$\begin{aligned} u &= \tan^{-1}x & dv &= x\,dx\\ du &= \frac{1}{x^2+1}\,dx & v &= \frac{1}{2}x^2 \end{aligned}$$

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[10] Some texts leave out the "$x =$" parts, assuming they are understood, but we will continue to use the convention that, unless otherwise stated, the "limits of integration" should match the differential's variable. Another popular way to write (7.18) avoids the issue:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\bigg|_a^b - \int_a^b v(x)u'(x)\,dx.$$

$$\begin{aligned}(\mathcal{I}) &= uv - \int v\,du \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx.\end{aligned}$$

Now this last integral can be found by rewriting the integrand using either polynomial long division, or by a little cleverness:

$$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} = 1 - \frac{1}{x^2+1}.$$

Thus

$$\begin{aligned}(\mathcal{I}) &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C.\end{aligned}$$

Though our choice of u *and* dv *was limited, our choice of* v *was not as limited. Recall that we could have chosen any* $v = \frac{1}{2}x^2 + C_1$*. In this particular integral, we could have saved ourselves some effort if we had chosen* v *more strategically:*

$$\begin{aligned} u &= \tan^{-1}x & dv &= x\,dx \\ du &= \frac{1}{x^2+1}\,dx & v &= \frac{1}{2}\left(x^2+1\right)\end{aligned}$$

This gives

$$\begin{aligned}(\mathcal{I}) &= uv - \int v\,du \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \int \frac{\frac{1}{2}(x^2+1)}{x^2+1}\,dx \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \int \frac{1}{2}\,dx \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \frac{1}{2}x + C,\end{aligned}$$

as before (though rearranged).

Though rare, and not crucial, strategically adding a particular constant to the natural choice for v can on occasion make for easier computations.

Exercises

Compute the following integrals, all of which can be computed “by parts.”

1. $\displaystyle\int x\sin x\,dx$

2. $\displaystyle\int x^2\sin x\,dx$

3. $\displaystyle\int x\cos x\,dx$

4. $\displaystyle\int x\sec x\tan x\,dx$

5. $\displaystyle\int x\sec^2 x\,dx$

6. $\displaystyle\int x\ln x\,dx$

7. $\displaystyle\int x\tan^{-1} x\,dx$

8. $\displaystyle\int x\sec^{-1} x\,dx,\ x>1$

9. $\displaystyle\int x\sec^{-1} x\,dx,\ x<1$

10. $\displaystyle\int x\sqrt{1-x}\,dx.$ (Parts optional)

11. $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$ (Parts optional)

12. $\displaystyle\int xe^x\,dx$

13. $\displaystyle\int x\sin 5x\,dx$

14. $\displaystyle\int xe^{x/2}\,dx$

15. $\displaystyle\int x^3e^{x^2}\,dx$

16. $\displaystyle\int x^5\sin x^3\,dx$

17. $\displaystyle\int x^2e^{3x}\,dx$

18. $\displaystyle\int \ln x\,dx$

19. $\displaystyle\int \tan^{-1} x\,dx$

20. $\displaystyle\int \sin^{-1} x\,dx$

21. $\displaystyle\int x\sqrt{1-x^2}\ \sin^{-1} x\ dx$

22. $\displaystyle\int x^3\sin 2x\,dx$

23. $\displaystyle\int \sin^2 x\,dx$ (Parts optional)

24. $\displaystyle\int \cos^2 5x\,dx$ (Parts optional)

25. $\displaystyle\int e^{5x}\cos 2x\,dx$

26. $\displaystyle\int \sec^3 x\,dx$

7.3 Trigonometric Integrals

We have already looked at two basic types of trigonometric integrals: those arising from the derivatives of the trigonometric functions (Subsection 6.1.4, page 408), and those of the elementary substitution types in Section 6.5. In this section we are mainly interested in computing integrals where the integrands are combinations of powers of trigonometric functions. In such cases, the angles of each trigonometric function appearing are all the same. Another important topic considered here is how to deal with trigonometric combinations where the angles differ, and we will examine how to deal with several of those cases.

In the first examples where the angles agree, we rearrange the terms in the integrand and use the three basic trigonometric identities to write the entire integral as function of one trigonometric function, and its differential as the final factor. A substitution step then leads to one or more power rules. Unfortunately this only leads to a solution if the combinations of powers are of a few simple forms. Still, these combinations occur often enough to warrant study.

After we look at those simplest forms, we look at other combinations of powers where the angles agree. Techniques include other algebraic manipulations, as well as integration by parts.

In the final forms, where the angles do not agree, we look at several trigonometric identities which help us to rewrite the integrals in simpler forms.

7.3.1 Simplest Examples

These first three examples illustrate an approach we develop in Subsections 7.3.2, 7.3.3 and 7.3.4.

Example 7.3.1 *Compute* $\int \tan^2 x\,dx$.

<u>Solution</u>: $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$

The example above used the facts that $\tan^2 x = \sec^2 x - 1$, and that we know the antiderivative of $\sec^2 x$ (where we might not have known the antiderivative of $\tan^2 x$ immediately). The integral above does not in itself contain a general method. Indeed there is no general method, but there are ways to rewrite many trigonometric integrals to make their computations more elementary.

Example 7.3.2 *Compute* $\int \sin^6 x\,\cos^5 x\,dx$.

<u>Solution</u>: *Here we will use the fact that* $\cos^2 x = 1 - \sin^2 x$, *and so* $\cos^{2k} x = \left(1 - \sin^2 x\right)^k$. *Eventually we will take* $u = \sin x$, *implying* $du = \cos x\,dx$:

$$\begin{aligned}
\int \sin^6 x\,\cos^5 x\,dx &= \int \sin^6 x\,\cos^4 x\,\cos x\,dx \\
\left.\begin{matrix} u = \sin x \\ du = \cos x\,dx \end{matrix}\right\} \qquad &= \int \sin^6 x\left(1 - \sin^2 x\right)^2 \cos x\,dx \\
&= \int u^6\left(1 - u^2\right)^2\,du = \int u^6\left(1 - 2u^2 + u^4\right)\,du \\
&= \int \left[u^6 - 2u^8 + u^{10}\right]\,du \\
&= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C \\
&= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + C.
\end{aligned}$$

Example 7.3.3 *Compute* $\int \sec^5 x\, \tan^3 x\, dx$.

Solution: Here we will borrow a factor of secant, and another of tangent, to form the functional part of du, *where* $u = \sec x$*:*

$$\begin{aligned}\int \sec^5 x\, \tan^3 x\, dx &= \int \sec^4 x\, \tan^2 x\, \sec x \tan x\, dx\\ &= \int \sec^4 x\left(\sec^2 x - 1\right)\sec x \tan x\, dx\\ &= \int u^4\left(u^2-1\right)\, du\\ &= \int \left[u^6 - u^4\right]\, du\\ &= \frac{u^7}{7} - \frac{u^5}{5} + C\\ &= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.\end{aligned}$$

Now we look at these three specific techniques more closely and generalize them.

7.3.2 Odd Powers of Sine or Cosine

Here we are interested in the cases of integrals

$$\int \sin^m \theta\, \cos^n \theta\, d\theta. \tag{7.19}$$

where either m or n is odd. Suppose, for example, that m is odd, so that we can write $m = 2k+1$ for some integer k. Then we rewrite the form (7.19) as

$$\int \sin^m \theta\, \cos^{2k+1} \theta\, d\theta = \int \sin^m \theta\, \cos^{2k} \theta\, \cos\theta\, d\theta.$$

The $\cos\theta$ term which we "peeled away" becomes the functional part of the du, where $u = \sin\theta$ (so $du = \cos\theta\, d\theta$). We then write the rest of the integral in terms of $u = \sin\theta$. To do so we use

$$\begin{aligned}&\sin^2\theta + \cos^2\theta = 0\\ \Longleftrightarrow\qquad &\cos^2\theta = 1 - \sin^2\theta\\ \Longrightarrow\qquad &\cos^{2k}\theta = \left(1-\sin^2\theta\right)^k.\end{aligned}$$

Using this fact in the integral above, and setting $u = \sin\theta$, we get

$$\begin{aligned}\int \sin^m \theta\, \cos^{2k+1} \theta\, d\theta &= \int \sin^m \theta\, \cos^{2k} \theta\, \cos\theta\, d\theta\\ &= \int \sin^m\theta\left(1-\sin^2\theta\right)^k\cos\theta\, d\theta\\ &= \int u^m\left(1-u^2\right)^k\, du.\end{aligned}$$

This yields a polynomial integrand, which we may then wish to expand before computing (with a sequence of power rules).

Simliarly, if there is an odd power of the sine, we can use the fact that $\sin^2\theta = 1-\cos^2\theta$, and eventually using $u=\cos\theta$, to rewrite such an integral

$$\begin{aligned}\int \sin^{2k+1}\theta\,\cos\theta\,d\theta &= \int \sin^{2k}\theta\,\cos\theta\sin\theta\,d\theta\\ &= \int \left(1-\cos^2\theta\right)^k\cos^n\theta\,\sin\theta\,d\theta\\ &= \int \left(1-u^2\right)u^n(-du)\\ &= -\int \left(1-u^2\right)u^n\,du.\end{aligned}$$

In both of these it was crucial that we had an odd number of factors of either the sine or cosine, since "peeling off" one factor then leaves an even number, which can be easily written in terms of the other trigonometric function. The peeled off factor is then the functional part of the differential after substitution.

Note that while any even power of a sine or cosine function can be written entirely in terms of the other, this is not the case with odd powers.[11] This technique works because removing a factor from an odd power of sine or cosine, both provides the functional part of du and leaves an even power, which we write in terms of the other function which is then u in the substitution.

Example 7.3.4 *Compute* $\int \sin^5 x\,\cos^4 x\,dx$.

Solution: Here we see an odd number of sine factors, as so we peel one away to be part of the differential term, and write the entire integral in terms of the cosine:

$$\begin{aligned}\int \sin^5 x\,\cos^4 x\,dx &= \int \sin^4 x\,\cos^4 x\,\sin x\,dx\\ &= \int \left(\sin^2 x\right)^2\cos^4 x\,\sin x\,dx\\ &= \int \left(1-\cos^2 x\right)^2\cos^4 x\,\sin x\,dx.\end{aligned}$$

(In most future computations we will skip the second line above.) Now we take

$$\begin{aligned} u &= \cos x\\ \Longrightarrow\quad du &= -\sin x\,dx\\ \Longleftrightarrow -du &= \sin x\,dx.\end{aligned}$$

With the substitution we will have a polynomial to integrate. To summarize and finish the

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[11] Consider the trigonometric identity $\sin^2\theta+\cos^2\theta=1$. When solved for either the sine or cosine function, we get one of the following:

$$\begin{aligned}\sin\theta &= \pm\sqrt{1-\cos^2\theta},\\ \cos\theta &= \pm\sqrt{1-\sin^2\theta}.\end{aligned}$$

We see the ambiguity in the ±, and the introduction of a radical which itself can very much complicate an integral. However, when we raise these to even powers the radicals *and* the ± both disappear, and we are left with sums of nonnegative, integer powers.

problem, we have:

$$\begin{aligned}
\int \sin^5 x\,\cos^4 x\,dx &= \int \left(1-\cos^2 x\right)^2 \cos^4 x\,\sin x\,dx \\
&= \int \left(1-u^2\right)^2 u^4(-du) \\
&= -\int \left(1-2u^2+u^4\right) u^4\,du \\
&= -\int \left(u^4-2u^6+u^8\right)\,du \\
&= -\frac{1}{5}u^5+\frac{2}{7}u^7-\frac{1}{9}u^9+C \\
&= -\frac{1}{5}\cos^5 x+\frac{2}{7}\cos^7 x-\frac{1}{9}\cos^9 x+C.
\end{aligned}$$

It should be clear that one cannot easily differentiate the final answer and immediately recognize the original integrand. This is because some trigonometric identities were used to get an integrand form which was computable using these methods. Indeed, it is best to check the validity of the steps from the beginning, rather than to differentiate a tentative answer. However, it is an interesting exercise—left to the interested reader—in trigonometric identities to perform the differentiation, and then validate that the answer there is the original integrand.

It is not necessary that the angle is always x. However, for this technique we do require the angles inside the trigonometric functions to always match, and for the approximate differential of the variable of substitution to be present.

Example 7.3.5 *Compute* $\int \sin^4 5x\,\cos^3 5x\,dx$.

<u>Solution</u>: Here there is an odd number of cosine terms, and we act accordingly.

$$\begin{aligned}
\int \sin^4 5x\,\cos^3 5x\,dx &= \int \sin^4 5x\,\cos^2 5x\,\cos 5x\,dx \\
&= \int \sin^4 5x\left(1-\sin^2 5x\right)\cos 5x\,dx.
\end{aligned}$$

Here we have

$$\begin{aligned}
& u = \sin 5x \\
\Longrightarrow\quad & du = 5\cos 5x\,dx \\
\Longleftrightarrow\quad & \frac{1}{5}\,du = \cos 5x\,dx.
\end{aligned}$$

Now we begin again, incorporating this new information into our computation:

$$\begin{aligned}
\int \sin^4 5x\,\cos^3 5x\,dx &= \int \sin^4 5x\left(1-\sin^2 5x\right)\cos 5x\,dx \\
&= \int u^4\left(1-u^2\right)\cdot\frac{1}{5}\,du \\
&= \frac{1}{5}\int \left(u^4-u^6\right)\,du \\
&= \frac{1}{5}\cdot\frac{1}{5}u^5-\frac{1}{5}\cdot\frac{1}{7}u^7+C \\
&= \frac{1}{25}\sin^5 5x-\frac{1}{35}\sin^7 5x+C.
\end{aligned}$$

The technique works even if only one of the trigonometric functions sine or cosine appears, as long as it is to an odd power.

Example 7.3.6 *Compute* $\int \sin^3 7x\,dx$.

Solution: Here we can still peel off a sine factor to be the functional part of our differential, and then write the remaining factors in terms of the cosine.

$$\begin{aligned}\int \sin^3 7x\,dx &= \int \sin^2 7x\ \sin 7x\,dx\\ &= \int \left(1-\cos^2 7x\right)\sin 7x\,dx.\end{aligned}$$

Using the substitution $u = \cos 7x$*, so* $du = -7\sin 7x\,dx$*, implying* $-\frac{1}{7}\,du = \sin 7x\,dx$*, we get*

$$\begin{aligned}\int \sin^3 7x\,dx &= \int \left(1-\cos^2 7x\right)\sin 7x\,dx\\ &= \int \left(1-u^2\right)\cdot\frac{-1}{7}\,du\\ &= -\frac{1}{7}\left[u-\frac{1}{3}u^3\right]+C\\ &= -\frac{1}{7}\cos 7x+\frac{1}{21}\cos^3 7x+C.\end{aligned}$$

Furthermore, not all the powers need to be positive integer powers, as long as one is odd.

Example 7.3.7 *Compute* $\int \frac{\cos^7 x}{\sqrt{\sin x}}\,dx$.

Solution: Here we have an odd number of cosine terms, so we will peel one off to be the functional part of our differential. That is, we will have $u = \sin x$*, so* $du = \cos x\,dx$*. Thus*

$$\begin{aligned}\int \frac{\cos^7 x}{\sqrt{\sin x}}\,dx &= \int \frac{\cos^6 x}{\sqrt{\sin x}}\ \cos x\,dx\\ &= \int \frac{\left(1-\sin^2 x\right)^3}{\sqrt{\sin x}}\ \cos x\,dx\\ &= \int \frac{\left(1-u^2\right)^3}{\sqrt{u}}\,du\\ &= \int \frac{1-3u^2+3u^4-u^6}{u^{1/2}}\,du\\ &= \int \left[u^{-1/2}-3u^{3/2}+3u^{7/2}-u^{11/2}\right]\,du\\ &= 2u^{1/2}-3\cdot\frac{2}{5}u^{5/2}+3\cdot\frac{2}{9}u^{9/2}-\frac{2}{13}u^{13/2}+C\\ &= 2u^{1/2}\left[1-\frac{3}{5}u^2+\frac{1}{3}u^4-\frac{1}{13}u^6\right]+C\\ &= 2\sqrt{\sin x}\left[1-\frac{3}{5}\sin^2 x+\frac{1}{3}\sin^4 x-\frac{1}{13}\sin^6 x\right]+C.\end{aligned}$$

It is possible that both powers are odd, and either function can be peeled off, and the integral written in terms of the other. However, if one of these odd powers is greater than the other, it is more efficient to peel off a factor from the lower power, as the next example demonstrates.

Example 7.3.8 *Compute* $\int \sin^3 x \cos^7 x\,dx$.

Solution: We will consider both methods for computing this antiderivative. First we peel off a sine to be part of the differential, and let $u = \cos x$.

$$\begin{aligned}\int \sin^3 x \cos^7 x\,dx &= \int \sin^2 x \cos^7 x \sin x\,dx\\ &= \int \left(1-\cos^2 x\right)\cos^7 x \sin x\,dx\\ &= \int \left(1-u^2\right)u^7(-\,du)\\ &= -\int \left(u^7-u^9\right)\,du\\ &= -\frac{1}{8}u^8+\frac{1}{10}u^{10}+C\\ &= -\frac{1}{8}\cos^8 x+\frac{1}{10}\cos^{10} x+C.\end{aligned}$$

Next we instead peel off a cosine factor, and let $w = \sin x$.

$$\begin{aligned}\int \sin^3 x \cos^7 x\,dx &= \int \sin^3 x \cos^6 x \cos x\,dx\\ &= \int \sin^3 x\left(1-\sin^2 x\right)^3 \cos x\,dx\\ &= \int w^3\left(1-w^2\right)^3\,dw\\ &= \int w^3\left(1-3w^2+3w^4-w^6\right)\,dw\\ &= \int \left(w^3-3w^5+3w^7-w^9\right)\,dw\\ &= \frac{1}{4}w^4-\frac{3}{6}w^6+\frac{3}{8}w^8-\frac{1}{10}w^{10}+C\\ &= \frac{1}{4}\sin^4 x-\frac{1}{2}\sin^6 x+\frac{3}{8}\sin^8 x-\frac{1}{10}\sin^{10} x+C.\end{aligned}$$

As we see in the above example, there can be different valid choices for some integrals. The answers may look very different, but that is a reflection of the wealth of trigonometric identities available. In fact, the antiderivatives, excluding the arbitrary constants, need not be equal, but the difference should be accounted for in the constants.[12]

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[12] Recall the integral $\int 2\sin x\cos x\,dx$, for which one can let either $u=\sin x$ or $u=\cos x$, yielding

$$\int 2\sin x\cos x\,dx = \sin^2 x + C_1, \qquad \text{or}$$
$$\int 2\sin x\cos x\,dx = -\cos^2 x + C_2.$$

Since these differ by a constant, specifically $\sin^2 x = -\cos^2 x + 1$, both are valid. But clearly $\sin^2 x \neq \cos^2 x$.

7.3.3 Even Powers of Secant or Odd Powers of Tangent

This technique of peeling off some factors of a trigonometric function to be part of the du (after substitution) has two workable versions for integrals of the type

$$\int \sec^m \theta \tan^n \theta \, d\theta. \tag{7.20}$$

These rely upon the following facts from trigonometry and calculus:

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta, & \frac{d}{d\theta}\tan\theta &= \sec^2\theta, \\ \sec^2\theta - 1 &= \tan^2\theta, & \frac{d}{d\theta}\sec\theta &= \sec\theta\tan\theta. \end{aligned}$$

The techniques we will employ are as follow:

1. If an integral of the form (7.20) contains an odd power of tangent, we peel off a factor $\sec\theta\tan\theta$ to be the functional part of the differential. This leaves an even power of tangent, which can be written as a power of $\left(\sec^2\theta - 1\right)$.
2. If an integral of the form (7.20) contains an even power of secant, we peel off a factor $\sec^2\theta$ to be the functional part of the differential. The remaining even power of secant is then written as a power of $\left(\tan^2\theta + 1\right)$.

Example 7.3.9 *Compute* $\int \sec^6 x \, \tan^8 x \, dx$.

Solution Here we have an even number of secant factors, and so we can peel off two. Eventually we will let $u = \tan x$, *implying* $du = \sec^2 x \, dx$.

$$\begin{aligned} \int \sec^6 x \, \tan^8 x \, dx &= \int \sec^4 x \, \tan^8 x \, \sec^2 x \, dx \\ &= \int \left(\tan^2 x + 1\right)^2 \tan^8 x \, \sec^2 x \, dx \\ &= \int \left(u^2+1\right)^2 u^8 \, du \\ &= \int \left(u^4 + 2u^2 + 1\right) u^8 \, du \\ &= \int \left(u^{12} + 2u^{10} + u^8\right) \, du \\ &= \frac{1}{13}u^{13} + \frac{2}{11}u^{11} + \frac{1}{9}u^9 + C \\ &= \frac{1}{13}\tan^{13} x + \frac{2}{11}\tan^{11} x + \frac{1}{9}\tan^9 x + C. \end{aligned}$$

Example 7.3.10 *Compute* $\int \sec^7 2x \tan^5 2x \, dx$

Solution: Here we have an odd number of tangent factors, so we peel off a $\sec 2x \tan 2x$ *factor to be the functional part of the differential. Eventually we then have* $u = \sec 2x$, *giving*

$du = 2\sec 2x\tan 2x\,dx$ and thus $\frac{1}{2}\,du = \sec 2x\tan 2x\,dx$.

$$\begin{aligned}\int \sec^7 2x\,\tan^5 2x\,dx &= \int \sec^6 2x\,\tan^4 2x\,\sec 2x\tan 2x\,dx\\ &= \int \sec^6 2x\left(\sec^2 2x-1\right)^2\sec 2x\tan 2x\,dx\\ &= \int u^6\left(u^2-1\right)^2\cdot\frac{1}{2}\,du\\ &= \frac{1}{2}\int u^6\left(u^4-2u^2+1\right)\,du\\ &= \frac{1}{2}\int\left(u^{10}-2u^8+u^6\right)\,du\\ &= \frac{1}{2}\left[\frac{1}{11}u^{11}-\frac{2}{9}u^9+\frac{1}{7}u^7\right]+C\\ &= \frac{1}{22}\sec^{11}2x-\frac{1}{9}\sec^9 2x+\frac{1}{14}\sec^7 2x+C.\end{aligned}$$

In fact this last example could be computed by first rewriting the integral in terms of cosines and sines:

$$\begin{aligned}\int\frac{\sin^5 2x}{\cos^{12}2x}\,dx &= \int\frac{\sin^4 2x}{\cos^{12}2x}\sin 2x\,dx = \int\frac{\left(1-\cos^2 2x\right)^2}{\cos^{12}2x}\sin 2x\,dx\\ &= \int\frac{\left(1-u^2\right)^2}{u^{12}}\cdot\frac{-1}{2}\,du = -\frac{1}{2}\int u^{-12}\left(1-2u^2+u^4\right)\,du,\ \text{etc.}\end{aligned}$$

Thus, the relationships involving the secant and tangent are not required in this last example. However, rewriting the integral in the previous problem, Example 7.3.9, in terms of sines and cosines would not yield either to an odd power. Thus Example 7.3.9 illustrates an integral which does benefit from the extra structure (algebraic and calculus) of the secant-tangent relationship.

Example 7.3.11 *Compute $\int \tan^4 x\,dx$.*

<u>Solution</u>: Here we look at two solutions. In the first, instead of exploiting the fact that there are an even number of factors of secant (namely zero) present here, we will repeatedly use the fact that $\tan^2\theta+1=\sec^2\theta$. (In the second line, we let $u=\tan x$.)

$$\begin{aligned}\int\tan^4 x\,dx &= \int\tan^2 x\left(sec^2x-1\right)\,dx\\ &= \int\underbrace{\tan^2 x}_{u^2}\underbrace{\sec^2 x}_{du}\,dx-\int\tan^2 x\,dx\\ &= \frac{1}{3}\tan^3 x-\int\tan^2 x\,dx\\ &= \frac{1}{3}\tan^3 x-\int\left(\sec^2 x-1\right)\,dx\\ &= \frac{1}{3}\tan^3 x-\tan x+x+C.\end{aligned}$$

Of course the other method is to "peel off" a factor of $\sec^2 x$, which we do even though it does not really appear. To have it appear, we will multiply and divide the integrand by $\sec^2 x$. Then *we will let* $u = \tan x$. *A long division will give us the sum of powers in our final integral below.*

$$\begin{aligned}\int \tan^4 x\,dx &= \int \frac{\tan^4 x}{\sec^2 x}\sec^2 x\,dx\\ &= \int \frac{\tan^4 x}{\tan^2 x+1}\sec^2 x\,dx\\ &= \int \frac{u^4}{u^2+1}\,du\\ &= \int \left(u^2-1+\frac{1}{u^2+1}\right)du\\ &= \frac{1}{3}u^3-u+\tan^{-1}u+C_1\\ &= \frac{1}{3}\tan^3 x-\tan x+\tan^{-1}(\tan x)+C_1\\ &= \frac{1}{3}\tan^3 x-\tan x+x+C.\end{aligned}$$

Here we did have the extra complication of long division. Furthermore, to see that the two answers were the same we had to notice $\tan^{-1}(\tan x) = x + n\pi$, *where* $n \in \mathbb{Z}$, *i.e.,* n *is an integer.*[13] *Thus the final constant* C *takes into account* $C_1 - n\pi$, *still a constant.*

7.3.4 Even Powers of Cosecant or Odd Powers of Cotangent

Here we just point out that a similar relationship exists between the cosecant and cotangent, as exists between the secant and tangent. We briefly look at two examples to illustrate this. The integral type is

$$\int \csc^m\theta\,\cot^n\theta\,d\theta. \tag{7.21}$$

We begin with the following facts from trigonometry and calculus:

$$\begin{aligned}\cot^2\theta+1&=\csc^2\theta, & \frac{d}{d\theta}\cot\theta&=-\csc^2\theta,\\ \csc^2\theta-1&=\cot^2\theta, & \frac{d}{d\theta}\csc\theta&=-\csc\theta\cot\theta.\end{aligned}$$

The techniques we will employ mirror those used for the secant-tangent integrals:

1. If an integral of the form (7.21) contains an odd power of cotangent, we peel off a factor $\csc\theta\cot\theta$ to be the functional part of the differential. This leaves an even power of cotangent, which can be written as a power of $\left(\csc^2\theta-1\right)$.
2. If an integral of the form (7.21) contains an even power of cosecant, we peel off a factor $\csc^2\theta$ to be the functional part of the differential. The remaining even power of cosecant is then written as a power of $\left(\cot^2\theta+1\right)$.

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[13] Recall knowing $\tan x$ does not mean we know the angle x, but we do know it to an integer multiple of π, which is the period of tangent. Recall also that tangent is one-to-one in each such period. (Of course the integral is only defined on each period individually, separated by the discontinuities—in fact vertical asymptotes—of the integrand.) Furthermore, the arctangent function only outputs angles in the period $-\pi/2 < \theta < \pi/2$, so $\tan^{-1}(\tan x) \in (-\pi/2, \pi/2)$, where x is not so restricted. Eventually, this is all taken care of by the arbitrary nature of the constant C.

Example 7.3.12 *Compute* $\displaystyle\int \csc^8 x\, \cot^2 x\, dx$.

<u>Solution</u>: We see an even number of cosecants, so we peel off two to be part of the differential.

$$\begin{aligned}\int \csc^8 x\, \cot^2 x\, dx &= \int \csc^6 x\, \cot^2 x\, \csc^2 x\, dx\\ &= \int \left(\csc^2 x\right)^3 \cot^2 x\, \csc^2 x\, dx\\ &= \int \left(\cot^2 x + 1\right)^3 \cot^2 x\, \csc^2 x\, dx.\end{aligned}$$

Taking $u = \cot x$, *giving* $du = -\csc^2 x\, dx$, *so* $-du = \csc^2 x\, dx$, *we get*

$$\begin{aligned}\int \csc^8 x\, \cot^2 x\, dx \int \left(\cot^2 x + 1\right)^3 \cot^2 x\, \csc^2 x\, dx\\ &= \int \left(u^2 + 1\right)^3 u^2(-du)\\ &= -\int \left(u^6 + 3u^4 + 3u^2 + 1\right) u^2\, du\\ &= -\int \left(u^8 + 3u^6 + 3u^4 + u^2\right)\, du\\ &= -\frac{1}{9}\,u^9 - \frac{3}{7}\,u^7 - \frac{3}{5}\,u^5 - \frac{1}{3}\,u^3 + C\\ &= -\frac{1}{9}\cot^9 x - \frac{3}{7}\cot^7 x - \frac{3}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C.\end{aligned}$$

Example 7.3.13 *Compute* $\displaystyle\int \csc^3 \frac{x}{2}\, \cot^3 \frac{x}{2}\, dx$.

<u>Solution</u>: Here the cotangent appears to an odd power, so we will peel of one cosecant and one cotangent.

$$\begin{aligned}\int \csc^3 \frac{x}{2}\, \cot^3 \frac{x}{2}\, dx &= \int \csc^2 \frac{x}{2}\, \cot^2 \frac{x}{2}\, \csc \frac{x}{2}\, \cot \frac{x}{2}\, dx\\ &= \int \csc^2 \frac{x}{2} \left(\csc^2 \frac{x}{2} - 1\right) \csc \frac{x}{2} \cot \frac{x}{2}\, dx.\end{aligned}$$

Now we let $u = \csc \frac{x}{2}$, *implying* $du = -\csc \frac{x}{2} \cot \frac{x}{2} \cdot \frac{1}{2}\, dx$, *whence* $-2\, du = \csc \frac{x}{2} \cot \frac{x}{2}\, dx$. *Our integral then becomes*

$$\begin{aligned}\int \csc^3 \frac{x}{2}\, \cot^3 \frac{x}{2}\, dx &= \int \csc^2 \frac{x}{2} \left(\csc^2 \frac{x}{2} - 1\right) \csc \frac{x}{2} \cot \frac{x}{2}\, dx\\ &= \int u^2 \left(u^2 - 1\right)(-2)\, du\\ &= -2\int \left(u^4 - u^2\right)\, du\\ &= -\frac{2}{5}\,u^5 + \frac{2}{3}\,u^3 + C\\ &= -\frac{2}{5}\csc^5 \frac{x}{2} + \frac{2}{3}\csc^3 \frac{x}{2} + C.\end{aligned}$$

7.3.5 Even Powers of Sine and Cosine

Now we turn our attention to the question of integration when both powers of sine and cosine are even. There are two standard methods for handling this: integration by parts, and "half-angle formulas." The former is more useful when the powers are small than when they are large, and the latter is perhaps more general.[14]

Example 7.3.14 *Compute* $\int \sin^2 x\,dx$ *using integration by parts.*

<u>Solution</u>: This exact computation was performed in Example 7.2.9, page 715. So that it is in front of us here, we summarize that computation:

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) &= \int \underbrace{\sin x}_{u}\underbrace{\sin x\,dx}_{dv} = \underbrace{(\sin x)}_{v}\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\underbrace{\cos x\,dx}_{du} = -\sin x\cos x + \int \cos^2 x\,dx\\ &= -\sin x\cos x + \int \left(1-\sin^2 x\right)\,dx = -\sin x\cos x + x - \int \sin^2 x\,dx\\ &= x - \sin x\cos x - (\boldsymbol{\mathcal{I}}).\end{aligned}$$

At this point we add $(\boldsymbol{\mathcal{I}}) = \int \sin^2 x\,dx$ *to both sides to get*

$$\begin{aligned}2\int \sin^2 x\,dx &= x - \sin x\cos x + C_1\\ \Longrightarrow \int \sin^2 x\,dx &= \frac{1}{2}\left(x-\sin x\cos x\right) + C.\end{aligned}$$

The method above works well for integrating $\sin^2 x$ or $\cos^2 x$, but higher, even powers become more cumbersome. For this reason it is common to opt for alternatives involving slightly more sophisticated trigonometric identities. There is some redundancy in the list below, as (7.22), (7.23), (7.24) and (7.26) together imply the others.

$$\begin{aligned}\sin(-\theta) &= -\sin\theta, && (7.22)\\ \cos(-\theta) &= \cos\theta, && (7.23)\\ \sin(A+B) &= \sin A\cos B + \sin B\cos A && (7.24)\\ \sin(A-B) &= \sin A\cos B - \sin B\cos A && (7.25)\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B && (7.26)\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B && (7.27)\\ \sin 2A &= 2\sin A\cos A && (7.28)\\ \cos 2A &= \cos^2 A - \sin^2 A && (7.29)\\ \cos 2A &= 2\cos^2 A - 1 && (7.30)\\ \cos 2A &= 1 - 2\sin^2 A. && (7.31)\end{aligned}$$

It is left for the exercises to show that

1. (7.25) follows from replacing B with $-B$ in (7.24),

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[14] In today's calculus texts, integration by parts is less prominently presented for such integrals, while half-angle methods are more popular among authors. We present both here for the lower powers, as some of the phenomena found in the integration by parts for such integrals are found later in this section.

2. Similarly, (7.27) follows from (7.26).
3. (7.28) follows from (7.24), if we let $B = A$.
4. Similarly (7.29) follows from (7.26).
5. (7.30) and (7.31) follow from (7.29) and the identity $\sin^2 A + \cos^2 A = 1$.

Now (7.30) and (7.31) can be rewritten as follow:

$$\begin{aligned}\cos 2A + 1 &= 2\cos^2 A,\\ 2\sin^2 A &= 1 - \cos 2A.\end{aligned}$$

Replacing A with θ, we can divide these by 2 to get so-called half-angle formulas:[15]

$$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right), \tag{7.32}$$

$$\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right). \tag{7.33}$$

Using (7.33), we see

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C.$$

For reasons which will be clear in the next section, it is often desirable that the angle in the final answer agree with the original angle, in this case x. For that we use the double-angle formula (7.28), to get

$$\begin{aligned}\int \sin^2 x\,dx &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C = \frac{1}{2}\left[x - \frac{1}{2}\cdot 2\sin x\cos x\right] + C\\ &= \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C.\end{aligned}$$

This agrees with the answer we obtained through integration by parts, in Example 7.3.14, page 729.

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[15] Equations (7.32) and (7.33) are called half-angle formulas because the angle θ on the left is half of the angle 2θ on the right. In fact, knowing the location of the terminal side of an angle does not tell us where its half is located. Indeed, 90° and 450° are coterminal, but their half angles, 45° and 225° are not. This is reflected in what are given as "half-angle" formulas in most trigonometry texts (compare to (7.32) and (7.33)):

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\ \sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}}.\end{aligned}$$

However, knowing where an angle terminates does determine where twice the angle terminates, as is reflected in (7.28)–(7.31).

Example 7.3.15 *Compute* $\int \sin^4 x\,dx$.

Solution: Here we use the half-angle formulas repeatedly, until our integral has no positive, even powers of sine or cosine:

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2\,dx = \int \left[\frac{1}{2}\left(1-\cos 2x\right)\right]^2\,dx\\ &= \frac{1}{4}\int \left(1-2\cos 2x+\cos^2 2x\right)\,dx\\ &= \frac{1}{4}\int \left[1-2\cos 2x+\frac{1}{2}(1+\cos 4x)\right]\,dx\\ &= \frac{1}{4}\int \left[\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right]\,dx\\ &= \frac{1}{4}\left[\frac{3}{2}\,x-\sin 2x+\frac{1}{2}\cdot\frac{1}{4}\sin 4x\right]+C\\ &= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.\end{aligned}$$

This answer is correct, but if we want to match the angles to the original (x), we can use some double-angle formulas (7.28) and (7.29):

$$\begin{aligned}\int \sin^4 x\,dx &= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C\\ &= \frac{3}{8}\,x-\frac{1}{4}\cdot 2\sin x\cos x+\frac{1}{32}\cdot 2\sin 2x\cos 2x+C\\ &= \frac{3}{8}\,x-\frac{1}{2}\sin x\cos x+\frac{1}{16}(2\sin x\cos x)(\cos^2 x-\sin^2 x)+C,\end{aligned}$$

which can again be simplified and rewritten in several ways.

Example 7.3.16 *Compute* $\int \sin^2 3x\,\cos^2 3x\,dx$.

Solution:

$$\begin{aligned}\int \sin^2 3x\,\cos^2 3x\,dx &= \int \frac{1}{2}(1-\cos 6x)\cdot\frac{1}{2}(1+\cos 6x)\,dx = \frac{1}{4}\int \left(1-\cos^2 6x\right)\,dx\\ &= \frac{1}{4}\int \left[1-\frac{1}{2}(1+\cos 12x)\right]\,dx = \frac{1}{4}\int \left[\frac{1}{2}-\frac{1}{2}\cos 12x\right]\,dx\\ &= \frac{1}{4}\left[\frac{x}{2}-\frac{1}{2}\cdot\frac{1}{12}\sin 12x\right]+C = \frac{x}{8}-\frac{1}{96}\sin 12x+C.\end{aligned}$$

(We could have used $1-\cos^2 6x=\sin^2 6x$ after the first line.) As before, if we would like to have our answer in terms of the original angle, we need to utilize the double angle formulas (7.28) and (7.29):

$$\begin{aligned}\int \sin^2 3x\,\cos^2 3x\,dx &= \frac{x}{8}-\frac{1}{96}\sin 12x+C\\ &= \frac{x}{8}-\frac{1}{96}\cdot 2\sin 6x\cos 6x+C\\ &= \frac{x}{8}-\frac{1}{48}(2\sin 3x\cos 3x)(\cos^2 3x-\sin^2 3x)+C\\ &= \frac{x}{8}-\frac{1}{24}\sin 3x\cos 3x(\cos^2 3x-\sin^2 3x)+C.\end{aligned}$$

7.3.6 Miscellaneous Problems and Methods-I

There are many trigonometric integrals that either do not fit one of the above categories, or for which those methods are unwieldy. We will look at several such here. The reader should realize, however, that we cannot exhaust all possibilities here, and a particular problem may have a particularly clever solution which does not generalize well to other problems.

The methods of the earlier subsections are all standard and any successful calculus student is expected to know them. The first few examples here are of this class also, in that the better students should be able to handle these without resorting to references. We will, however, eventually have methods in this subsection which such a student should be aware of, but is understandably less likely to be able to recite from memory. All are derivable, but again, the latter are somewhat more obscure and even an excellent students might prefer to use a reference. It is important, however, that all students be aware of these latter classes of problems, and the available methods of solution, regardless of whether a reference is used ultimately.

Example 7.3.17 *Compute* $\displaystyle\int \sec^3 x\,dx$.

Solution: *Here we have an odd number of secants, and an even (zero) number of tangents. Unfortunately our ealier methods called for an even number of secants or an odd number of tangents. We could notice that the integrand represents an odd number* (-3) *of cosines, and then with* $u = \sin x$, *we could write*

$$\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}\,dx = \int \frac{1}{\cos^4 x}\,\cos x\,dx = \int \frac{1}{\left(1-\sin^2 x\right)^2}\cos x\,dx = \int \frac{1}{(1-u^2)^2}\,du,$$

but in fact we have yet to discuss how to integrate that final form. (We will in Section 7.5, and while it will be somewhat long, it will be a straightforward computation there). Instead we will next try integration by parts. Since the integrand contains an easily integrated $\sec^2 x\,dx$ *factor, we will let that be* dv:

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x\,dx \\ du &= \sec x \tan x dx & v &= \tan x \end{aligned}$$

$$\int \sec^3 x\,dx = uv - \int v\,du = \sec x \tan x - \int \sec x \tan^2 x\,dx.$$

Now it is tempting to do another parts step, with $dv = \sec x \tan x\,dx$, *but—as happened in some previous examples—we would then have our original integral on the left, and the same on the right. What works here is to instead use one of the basic trigonometric identities at this step:*

$$\begin{aligned} \int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\ &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) + C \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. \end{aligned}$$

Adding $\int \sec^3 x\,dx$ *to both sides gives*

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x\tan x + \ln|\sec x+\tan x| + C_1 \\ \Longrightarrow \quad \int \sec^3 x\,dx &= \frac{1}{2}(\sec x\tan x + \ln|\sec x+\tan x|) + C. \end{aligned}$$

When integrating arbitrary integer powers of the trigonometric functions, a common technique is to make use of so-called *reduction formulas.* These are derived using integration by parts, often incorporating the kind of computation above. For instance, let us consider the general problem of integrating $\sec^n x$, where $n \geq 3$. Such an integral contains within its integrand the factor $\sec^2 x$, which we use in the dv term. Integration by parts can proceed as follows:

$$\begin{aligned} u &= \sec^{n-2} x & dv &= \sec^2 x\,dx \\ du &= (n-2)\sec^{n-3} x\ \sec x\cdot\tan x\,dx & v &= \tan x \\ \Longrightarrow \quad du &= (n-2)\sec^{n-2} x\ \tan x\,dx & & \end{aligned}$$

giving us

$$\begin{aligned} \int \sec^n x\,dx &= \int \underbrace{\sec^{n-2} x}_{u}\underbrace{\sec^2 x\,dx}_{dv} \\ &= \sec^{n-2} x\ \tan x - \int (n-2)\sec^{n-2} x\ \tan^2 x\,dx \\ &= \sec^{n-2} x\ \tan x - \int (n-2)\sec^{n-2} x\left(\sec^2 x - 1\right)\,dx \\ &= \sec^{n-2} x\ \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \\ \Longrightarrow \quad (n-1)\int \sec^n x\,dx &= \sec^{n-2} x\ \tan x + (n-2)\int \sec^{n-2} x\,dx. \end{aligned}$$

Now we can divide by $(n-1)$ to get a general reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\ \sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \tag{7.34}$$

This is called a reduction formula because the resulting integral is of a lower power of secant. A quick inspection reveals that this formula is valid for $n = 2$ as well, so it is if fact valid for $n \geq 2$.

Example 7.3.18 *Compute* $\displaystyle\int \sec^5 x\,dx$.

Solution: Here we will invoke the formula twice: once for $n = 5$*, and then again for* $n = 3$ *to deal with the resulting integral. That will give an integral of secant to the first power, which is one which should be already memorized.*

$$\begin{aligned} \int \sec^5 x\,dx &= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{4}\int \sec^3 x\,dx && \textit{(n=5 in (7.34))} \\ &= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{4}\left[\frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx\right] && \textit{(n=3 in (7.34))} \\ &= \frac{1}{4}\sec^3 x\ \tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x+\tan x| + C. \end{aligned}$$

Other reduction formulas which can be arrived at similarly include

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\,\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \tag{7.35}$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\,\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx. \tag{7.36}$$

Example 7.3.19 *Compute* $\int \cos^6 5x\,dx$.

Solution: Here we cannot use the formula (7.36) directly, because our angle does not match our differential. To compensate, we will perform a substitution step first. Specifically, we will let $u = 5x$, *so* $du = 5dx$ *and thus* $\frac{1}{5}\,du = dx$, *giving*

$$\begin{aligned}
\int \cos^6 5x\,dx &= \int \cos^6 u \cdot \frac{1}{5}\,du \\
&= \frac{1}{5}\int \cos^6 u\,du \\
&= \frac{1}{5}\left[\frac{\cos^5 u\,\sin u}{6} + \frac{5}{6}\int \cos^4 u\,du\right] && \textit{(n = 6 in (7.36))} \\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{1}{6}\left[\frac{\cos^3 u\,\sin u}{4} + \frac{3}{4}\int \cos^2 u\,du\right] && \textit{(n = 4 in (7.36))} \\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{\cos^3 u\,\sin u}{24} + \frac{1}{8}\left[\frac{\cos u \sin u}{2} + \frac{1}{2}\int 1\,du\right] && \textit{(n = 2 in (7.36))} \\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{\cos^3 u\,\sin u}{24} + \frac{\cos u \sin u}{16} + \frac{1}{16}u + C \\
&= \frac{\cos^5 5x\,\sin 5x}{30} + \frac{\cos^3 5x\,\sin 5x}{24} + \frac{\cos 5x \sin 5x}{16} + \frac{5x}{16} + C.
\end{aligned}$$

Clearly reduction formulas can be very useful. Indeed they provide an iterative method for reducing an integral computation, step by step, until—hopefully—a manageable integral appears. In fact in both examples above, we used the reduction formula for one more step than necessary, because it was easier than recomputing $\int \sec^3 x\,dx$ or $\int \cos^2 u\,du$ as before. Furthermore, many of the technical details for finding these integrals are built into the reduction formulas.

As useful as the reduction formulas are, they have a couple of minor drawbacks. First, when the angle does not match the differential, some substitution needs to be performed to compensate. Second—and more serious—is that any attempt to memorize these is likely to result in error. Thus the student of calculus needs to learn the earlier methods and be able to perform such calculations unaided, and also know that these reduction formulas (and others) are available and know how to use them.[16]

Now we consider integrals of following three forms, where $m \neq n$:

$$\int \sin mx\,\sin nx\,dx, \qquad \int \sin mx\,\cos nx\,dx, \qquad \int \cos mx\,\cos nx\,dx.$$

What distinguishes these is that the angles of the trigonometric functions do not agree. There are two methods for computing these: integration by parts, and utilizing the following trigonometric

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[16] Such formulas can be found in most engineering/science calculus texts, as well as books containing tables of integration formulas such as the *CRC Standard Mathematical Tables and Formulae.*

identities:

$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right], \tag{7.37}$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right], \tag{7.38}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]. \tag{7.39}$$

For obvious reasons, these are called *product-sum formulas.* These follow from adding or subtracting (7.24), (7.25), (7.26), and (7.27), which we repeat here for reference:

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\sin B\cos A,\\ \sin(A-B) &= \sin A\cos B-\sin B\cos A,\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B,\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B.\end{aligned}$$

For instance, (7.37) follows from adding the first two of these and solving for $\sin A\cos B$. All three, (7.37)–(7.39), are left to the exercises.

Example 7.3.20 *Compute* $\displaystyle\int \sin 2x\cos 5x\,dx$*.*

Solution: Here we use (7.37), with $A = 2x$ and $B = 5x$:

$$\begin{aligned}\int \sin 2x\cos 5x\,dx &= \int \frac{1}{2}[\sin(2x-5x)+\sin(2x+5x)]\,dx\\ &= \int \frac{1}{2}[\sin(-3x)+\sin 7x]\,dx\\ &= \int \frac{1}{2}[-\sin 3x+\sin 7x]\,dx\\ &= \frac{1}{3}\cos 3x-\frac{1}{7}\cos 7x+C.\end{aligned}$$

The method above has the drawback that the solution does not contain the same angles as the integrand. One can get back to the original angles using the formulas

$$\begin{aligned}\cos 3x &= \cos(5x-2x) = \cos 5x\cos 2x+\sin 5x\sin 2x,\\ \cos 7x &= \cos(5x+2x) = \cos 5x\cos 2x-\sin 5x\sin 2x.\end{aligned}$$

Alternatively, an integration by parts argument leaves intact the angles. It requires two integration by parts steps, and we need to solve for the integral. Furthermore, we have to make the analogous substitution for u both times, and for dv both times. By analogous, here we mean using the same angle, $2x$ or $5x$, as the argument of the trigonometric function both times. If we always let the u-term have angle $2x$, and the dv-term have angle $5x$, eventually the solution there will be

$$\int \sin 2x\cos 5x\,dx = \frac{5}{21}\sin 2x\sin 5x+\frac{2}{21}\cos 2x\cos 5x+C.$$

While it is far from obvious that these results are the same, it is an interesting exercise to check this last computation by computing the derivative of our answer here, so see how it simplifies to the integrand.

7.3.7 Miscellaneous Problems and Methods-II

It is often the case that trigonometric integrals arise in the process of using another technique, and so the form of the integral may be more awkward than we have illustrated here so far. This will be the case particularly in Section 7.4. In such cases it is often necessary to experiment with rewriting the integral using whatever trigonometric identities apply. One also has to be aware that a substitution or integration-by-parts argument may be required eventually. Even then, if it is a multistep application of integration by parts, it helps to recall when it helps to continue the "parts" step, and when it is better to use a trigonometric identity and solve for the integral algebraically, as when we integrated $\sec^3 x$ or $\csc^3 x$ (see Example 7.3.17, completed on page 733). In fact, once part of the process yields such an integral, it helps to know that such a complication was probably inevitable and we might as well deal with it from there, rather than attempt to bypass that problem.

With experience one learns to look ahead a few steps and anticipate which algebraic manipulation or identity will yield positive progress towards a form we can integrate (even if it requires some cleverness such as needed to integrate $\sec^3 x$), but it is not uncommon to require multiple attempts to integrate a trigonometric integral before finding a strategy that will ultimately succeed.

Example 7.3.21 *Compute* $\displaystyle\int \tan^2\theta\sec\theta\,d\theta$.

Solution: We will attempt to compute this integral two different methods. First we will integrate by parts, noting that

$$\int \tan^2\theta\sec\theta\,d\theta = \int \tan\theta\cdot\sec\theta\tan\theta\,d\theta.$$

$$\begin{aligned} u &= \tan\theta & dv &= \sec\theta\tan\theta\,d\theta \\ du &= \sec^2\theta\,d\theta & v &= \sec\theta \end{aligned}$$

$$\begin{aligned}
\int \tan^2\theta\sec\theta\,d\theta &= \int \underbrace{\tan\theta}_{u}\underbrace{\sec\theta\tan\theta\,d\theta}_{dv} \\
&= \underbrace{\tan\theta\sec\theta}_{uv} - \int \underbrace{\sec\theta}_{v}\underbrace{\sec^2\theta\,d\theta}_{du} \\
&= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \sec^2\theta\sec\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \left[\left(\tan^2\theta+1\right)\sec\theta\right]\,d\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \int \sec\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \ln|\sec\theta+\tan\theta| + C_1 \\
\Longrightarrow 2\int \tan^2\theta\sec\theta\,d\theta &= sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C_1 \\
\Longrightarrow \quad \int \tan^2\theta\sec\theta\,d\theta &= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

Because an intermediate step included the integral of $\sec^3\theta$*, which by itself would require an integration by parts technique that includes solving for the integral, it was likely that part of the solution would include our original integral on the left. Rather than beginning an integration by parts, the simple experiment of rewriting* $\sec^3\theta$ *as* $\sec^2\theta\sec\theta$*, and ultimately* $(\tan^2\theta+1)\sec\theta$*, was successful in producing a form in which we could solve for our original integral.*

Alternatively we could have computed $\int\sec^3\theta\,d\theta$ *elsewhere and inserted the computation here. That computation could have been carried out as before (Example 7.3.17), or through the appropriate reduction formula (7.34), page 733.*

The next example can be computed in two ways, but the first that we show here in fact requires a technique from a later section, namely Section 7.5. We show it here anyways in anticipation of the time when the student can call upon the ideas of more than one of these sections for a single problem, as we did previously (in using integration by parts to solve a trigonometric integral, for instance).

Example 7.3.22 *Compute* $\displaystyle\int\frac{\cos^4 x}{\sin x}\,dx$.

<u>*Solution 1:*</u> *Note that this integral does in fact contain an odd number of sine factors, even though that number is* -1*. We can therefore make the integral one in terms of cosine only as the integrand, with the differential containing the sine:*

$$\begin{aligned}\int\frac{\cos^4 x}{\sin x}\,dx&=\int\frac{\cos^4 x}{\sin^2 x}\cdot\sin x\,dx\\&=\int\frac{\cos^4 x}{1-\cos^2 x}\cdot\sin x\,dx.\end{aligned}$$

At this point, we have $u=\cos x\implies du=-\sin x\,dx\implies -du=\sin x\,dx$*, and then we eventually us some long division of polynomials to get*

$$\begin{aligned}\int\frac{\cos^4 x}{\sin x}\,dx&=\int\frac{\cos^4 x}{1-\cos^2 x}\cdot\sin x\,dx=-\int\frac{u^4}{1-u^2}\,du=\int\frac{u^4}{u^2-1}\,du\\&=\int\left[u^2+1+\frac{1}{u^2-1}\right]\,du.\end{aligned}$$

At this point we need a technique from the Partial Fractions section (Section 7.5) to expand the fraction still inside the integral to get[17]

$$\frac{1}{u^2-1}=-\frac{1}{2}\cdot\frac{1}{u+1}+\frac{1}{2}\cdot\frac{1}{u-1}.$$

While we mention only the final result of that computation, it is easily enough verified by combining the fractions on the right-hand side. To finish our integral computation, we would then write

$$\begin{aligned}\int\frac{\cos^4 x}{\sin x}\,dx&=\int\left[u^2+1-\frac{1/2}{u+1}+\frac{1/2}{u-1}\right]\,du\\&=\frac{1}{3}u^3+u-\frac{1}{2}\ln|u+1|+\frac{1}{2}|u-1|+C\\&=\frac{1}{3}\cos^3 x+\cos x-\frac{1}{2}\ln|\cos x+1|+\frac{1}{2}\ln|\cos x-1|+C.\end{aligned}$$

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[17] Here we could instead recall the derivative formula for $\tanh^{-1}x$ and use it for our antiderivative here, since $u=\cos x\in[-1,1]$:

$$\int\frac{1}{1-u^2}\,du=\tanh^{-1}u+C.$$

Solution 2: *An alternative approach is to rewrite the entire integral in terms of the sine function:*

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= \int \frac{\left(\cos^2 x\right)^2}{\sin x}\,dx = \int \frac{\left(1-\sin^2 x\right)^2}{\sin x}\,dx\\ &= \int \frac{1-2\sin^2 x+\sin^4 x}{\sin x}\,dx = \int \left(\csc x - 2\sin x + \sin^3 x\right)\,dx\\ &= -\ln|\csc x+\cot x| + 2\cos x + \int \sin^3 x\,dx.\end{aligned}$$

This last integral is then one with an odd power of sine, so we use our previous techniques, with $u = \cos x \implies du = -\sin x\,dx$*:*

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x\cdot\sin x\,dx = \int \underbrace{\left(1-\cos^2 x\right)}_{1-u^2}\underbrace{\sin x\,dx}_{-du}\\ &= -\cos x + \frac{1}{3}\cos^3 x + C.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= -\ln|\csc x+\cot x| + 2\cos x - \cos x + \frac{1}{3}\cos^3 x + C\\ &= -\ln|\csc x+\cot x| + \cos x + \frac{1}{3}\cos^3 x + C.\end{aligned}$$

Exercises

Evaluate the following integrals.

1. $\displaystyle\int \sin x\cos x\,dx$
2. $\displaystyle\int \sin^2 x\cos x\,dx$
3. $\displaystyle\int \sin x\cos^2 x\,dx$
4. $\displaystyle\int \sin^3 x\cos^2 x\,dx$
5. $\displaystyle\int \sin^4 x\cos^5 x\,dx$
6. $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx$
7. $\displaystyle\int \frac{\sin^3 x}{\cos^2 x+1}\,dx$
8. $\displaystyle\int \sin^4 x\cos^5 x\,dx$
9. $\displaystyle\int \sin^3 2x\cos^{15} 2x\,dx$
10. $\displaystyle\int \frac{\sin^2 x}{\cos x}\,dx$
11. $\displaystyle\int \sin^3 x\ln|\sin x|\,dx$
12. $\displaystyle\int \cos^3 x\ln|\sin x|\,dx$

7.4 Trigonometric Substitution

In this section we explore how integrals can sometimes be solved by making some clever substitutions involving trigonometric functions, even though the original integrals themselves do not involve such functions.

7.4.1 Introduction

Before developing the general mechanics, we look at a few examples below for motivation.

Example 7.4.1 *Compute* $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$*, using a nontrivial substitution method.*

Solution: We already know the answer to this integral, because we know $\frac{d}{dx}\sin^{-1}x = 1/\sqrt{1-x^2}$*, so a "clever" but unnecessary substitution* $u = \sin^{-1}x$ *would yield the antiderivative quickly.*[18]

But let us imagine for a moment that we do not have this antiderivative at our disposal, and need to tackle this integral from first principles. Can we accomplish this without first developing a theory of derivatives of arc-trigonometric functions?

The answer is yes, if we are clever in a different way, which offers a more general method we can apply to a class of more complicated integrals, as we will see. This substitution method is to notice that $-1 < x < 1$ *is required in the integral, and that is nearly the same as the range of* $\sin\theta$*:* $-1 \le \sin\theta \le 1$*, and is in fact contained in it. So we make a substitution, albeit somewhat "backwards" from what we are used to, where* x *will be a function of* θ *rather than a function of* x *being explicitly some new variable:*

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{\cos\theta}{\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}x + C$$

$$\left.\begin{array}{r} x = \sin\theta \\ \Longrightarrow\ dx = \cos\theta\,d\theta \end{array}\right|$$

The above computation is completely correct, but there are a few technicalities to check.

1. *Why are we allowed to take the nonnegative case above, when we know* $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$*? In other words, usually* $\cos\theta = \pm\sqrt{1-\sin^2\theta}$*, but we took the "+" case.*
2. *Why can we automatically say* $x = \sin\theta \implies \theta = \sin^{-1}x$*? After all, there are infinitely many angles with the same sine, and they need not necessarily even be coterminal when we graph them in standard position. (Example:* $\sin 30° = \sin 150° = \sin 390°$*, etc.)*

The answer to both questions lies in the values for θ *that we can choose when we make the substitution. In fact we were negligent by not fixing the range of* θ *from the outset, but we*

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[18] Anytime we know the antiderivative, we can perform a substitution where we define u to be that antiderivative, and the resulting integral will be $\int du = u + C$, but this has relatively little use for discovery. For the integral in Example 7.4.1, we could have written

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int du = u + C = \sin^{-1}x + C.$$

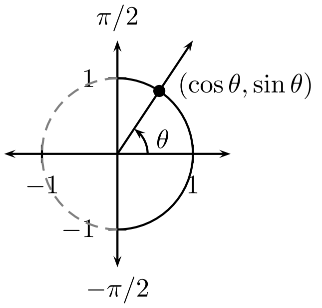
$$\left.\begin{array}{r} u = \sin^{-1}x \\ \Longrightarrow\ du = \dfrac{1}{1-x^2}\,dx \end{array}\right\}$$

will see there is a standard practice in which we will take exactly those angles θ *(or a subset of them) which are the same as the range of the relevant arc-trigonometric function. For the problem above, it is assumed* $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *since* $\sin\theta$ *is one-to-one for these values, and* $\sin\theta$ *ranges over the whole range of the sine funtion for these values, i.e., for the same reason the arcsine is defined to output these values.*

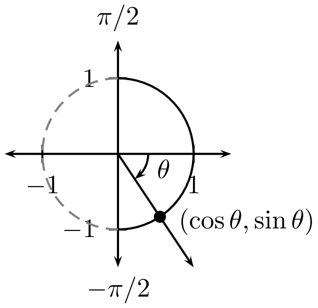
$$\begin{tikzcd} \underbrace{[-1,1]}_{\substack{\rotatebox{90}{$\in$}\\ x}} \arrow[r, mapsto, "\sin^{-1}x"] & \underbrace{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}_{\substack{\rotatebox{90}{$\in$}\\ \theta}} \arrow[r, mapsto, "\sin\theta"] & \underbrace{[-1,1]}_{\substack{\rotatebox{90}{$\in$}\\ x}} \end{tikzcd}$$

Thus we have mappings from x *to* θ *and back. When we actually compute the integral, we concentrate on the second mapping more than the first, until we compute the antiderivative, in which case we may need the first mapping to "get back" to* x.

When we look at the values of θ *and their terminal points on the unit circle, all doubt about our casual computation* $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ *is laid to rest, because cosine is nonnegative in our range of angles* θ*:*



$x = \sin\theta \geq 0$
$\theta \in [0, \pi/2]$
$\cos\theta \geq 0$



$x = \sin\theta \leq 0$
$\theta \in [-\pi/2, 0]$
$\cos\theta \geq 0$

Now "x*" here does not signify the coordinate on the horizontal axis. Indeed, for this particular case "*x*" is the same as* $\sin\theta$, *which is the vertical ("*y*") coordinate where our angle in standard position (initial side being the positive horizontal axis) pierces the unit circle.*

Note that the angles θ *ranging from* $-\pi/2$ *to* $\pi/2$ *is the same as* $\sin\theta$ *ranging from* -1 *to* 1, *and for such* θ *we indeed have*

1. $x = \sin\theta \iff \theta = \sin^{-1}x$, *and*
2. $\cos\theta \geq 0$, *so* $\sqrt{1-\sin^2\theta} = \cos\theta$.

This ends our first example.

The general outline for integrals which are appropriate for the technique is as follows:

1. Make a substitution to rewrite the integral in terms of an angle θ (with appropriate ranges of θ matching the range of values for x).
2. Compute the resultant trigonometric integral.
3. Rewrite the antiderivative in terms of x.

Drawing diagrams as above will be quite useful in subsequent problems. The process can not only clarify somewhat our substitution process, but it can also allow us to check that we have correct signs for all the various cases for θ. Moreover, there are times when we need to read actual values of *other* trigonometric functions of θ, but in terms of x.

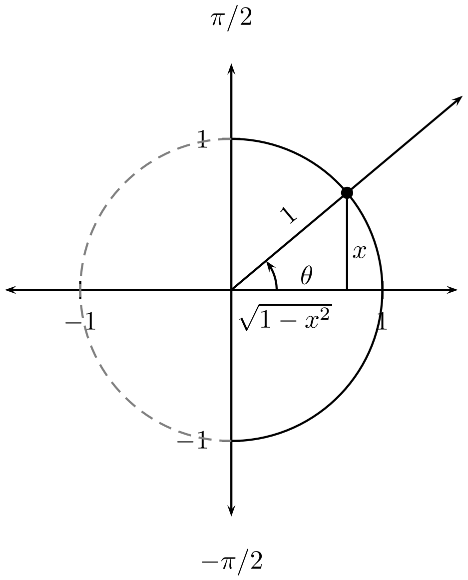
The process above may seem unnecessarily complicated, especially for an integral which for which we knew the answer from the beginning. However, this advanced technique generalizes to integrals which do not succumb to previous methods (though they should always be considered first!). We would be hard-pressed to compute the antiderivative in the next example using earlier techniques.

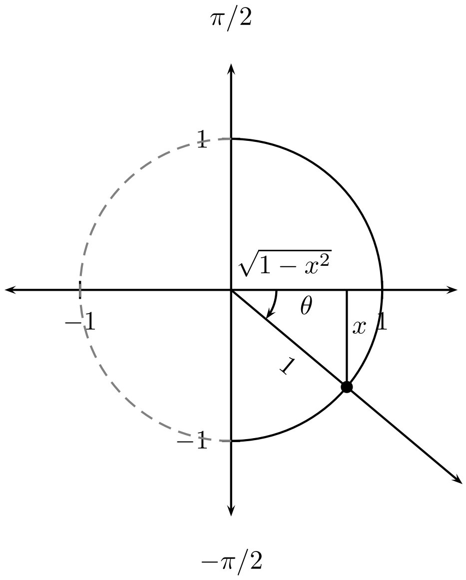
Example 7.4.2 *Compute* $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$.

Solution: Note first that this integral will not simply yield to earlier techniques. (The reader is welcome to try, to see where those methods eventually fall short.)

Note also that, due to the square root, we require $-1 \le x \le 1$. *In fact we also cannot have* $x = 0$, *but that constraint will be consistent with our new integral upon substitution. Indeed it is the radical which is giving us the most difficulty here.*

The solution to that problem is to again realize that $[-1, 1]$ *is exactly the range of the function* $\sin\theta$, *so we will again use the substitution* $x = \sin\theta$ *in the integral above, with the understanding that* $\theta \in [-\pi/2, \pi/2]$ *(excluding zero due to the denominator). This time we will again draw the diagram, but will label the various parts of the resuting triangles for future reference. Note that in the second drawing,* $\sin\theta = x < 0$ *while* $\cos\theta = \sqrt{1-x^2}$ *is still a positive quantity, and again, where the angle's terminal ray pierces the unit circle is the point* $(\cos\theta, \sin\theta)$.





Note that we can now read all trigonometric functions of θ *from the diagrams. For instance,* $\tan\theta = x/\sqrt{1-x^2}$, *regardless of which of the two quadrants contains the terminal ray of* θ.

Now we perform the substitution, noting as usual that the differential must also be accounted for:

$$\begin{aligned} x &= \sin\theta \\ \implies dx &= \cos\theta\,d\theta. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x}\,dx &= \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\,d\theta = \int \frac{\cos\theta}{\sin\theta}\cos\theta\,d\theta \\ &= \int \frac{\cos^2\theta}{\sin\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \int \left[\csc\theta - \sin\theta\right]\,d\theta \\ &= -\ln|\csc\theta - \cot\theta| + \cos\theta + C. \end{aligned}$$

Now we have a trigonometric form of the antiderivative, but of course the original integral was in x and not θ. Our diagram allows us to read the other trigonometric functions of θ in terms of x. We note that it does not matter, for this example, which of the two quadrants contains θ:

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x}\,dx &= -\ln|\csc\theta - \cot\theta| + \cos\theta + C = -\ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C \\ &= -\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C = \ln\left|\frac{x}{1-\sqrt{1-x^2}}\right| + \sqrt{1-x^2} + C \\ &= \ln|x| - \ln\left(1-\sqrt{1-x^2}\right) + \sqrt{1-x^2} + C. \end{aligned}$$

Note that the calculus was finished on the first line in the above equations, and the rest were algebraic rewritings. That absolute values were not needed in the first expression on the last line was due to the fact that $1 \ge \sqrt{1-x^2} \implies 1-\sqrt{1-x^2} \ge 0$.

Unlike the first example, we are not likely to be anxious to check our work by taking the derivative of our solution (though it would be an interesting exercise, particularly to see how terms cancel), so instead we strive to be careul and clear about our derivation, so we can minimize errors and easily re-read our computations to verify our result.

It should be noted at the outset that the trigonometric integrals which arise here may require some re-writing before they succumb to our trigonometric integral methods of Section 7.3; a problem which naturally gives rise to trigonometric substitution (as in the previous example) may or may not yield a simple trigonometric integral. However, all trigonometric integrals we will encounter here are of classes we considered in Section 7.3, so ultimately those techniques equipped us for our work here.

Before we delve into other trigonometric substitutions, we will perform one more involving the sine.

Example 7.4.3 *Compute* $\int \sqrt{9-25x^2}\,dx$.

Solution: As always, we should look to see if previous methods apply. They do not, without extraordinary cleverness, though it is interesting to note that simple substitution would work if we had an extra factor of x in the integrand. (We do not.)

Now in previous examples, we wanted to exploit the trigonometric identity $1-\sin^2\theta = \cos^2\theta$. Here we will do the same, except we will multiply this equation by 9, giving us $9 - 9\sin^2\theta = 9\cos^2\theta$.

So we wish to have $25x^2 = 9\sin^2\theta$, *i.e.,* $x^2 = \frac{9}{25}\sin^2\theta$. *We get this if we let* $x = \frac{3}{5}\sin\theta$.

$$\int\sqrt{9-25x^2}\,dx = \int\sqrt{9-25\cdot\frac{9}{25}\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta$$

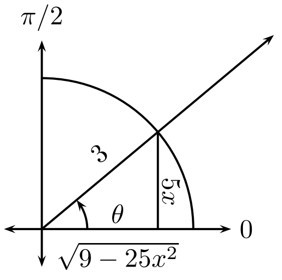
$$\left.\begin{aligned} 25x^2 &= 9\sin^2\theta \\ x^2 &= \frac{9}{25}\sin^2\theta \\ \therefore\ \text{Take } x &= \frac{3}{5}\sin\theta \\ \Longrightarrow\ dx &= \frac{3}{5}\cos\theta\,d\theta \end{aligned}\right|$$

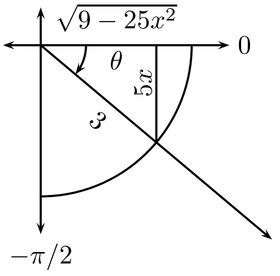
$$\begin{aligned}
&= \int\sqrt{9-9\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta \\
&= \int\sqrt{9(1-\sin^2\theta)}\cdot\frac{3}{5}\cos\theta\,d\theta \\
&= \int 3\sqrt{1-\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta \\
&= \int\frac{9}{5}\cos^2\theta\,d\theta \\
&= \int\frac{9}{5}\cdot\frac{1}{2}\,(1+\cos 2\theta)\;d\theta \\
&= \frac{9}{10}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C \\
&= \frac{9}{10}\theta+\frac{9}{10}\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta + C \\
&= \frac{9}{10}\theta+\frac{9}{10}\sin\theta\cos\theta + C.
\end{aligned}$$

We require the final trigonometric antiderivative to be in terms of the angle θ *(and not* 2θ *for example), so that we can read the trigonometric functions of* θ *from the diagram. In constructing the diagram, we need to solve our subsitution for* $\sin\theta$*:*

$$x = \frac{3}{5}\sin\theta \implies \sin\theta = \frac{5x}{3}.$$

Note that our diagram below is not constructed on the unit circle, but on a more general circle of positive radius (in this case 3). As before, we have to consider both quadrants in which θ *may fall.*



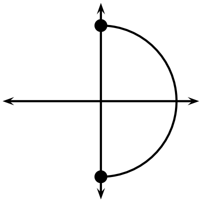
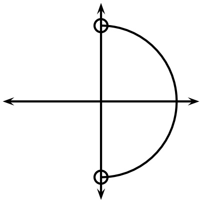
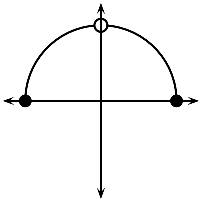


From these and our expression for $\sin\theta$ above we can complete our integral computation:

$$\begin{aligned}
\int\sqrt{9-25x^2}\,dx &= \frac{9}{10}\theta+\frac{9}{10}\sin\theta\cos\theta + C \\
&= \frac{9}{10}\sin^{-1}\left(\frac{5x}{3}\right)+\frac{9}{10}\cdot\frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}+C \\
&= \frac{9}{10}\sin^{-1}\left(\frac{5x}{3}\right)+\frac{x}{2}\sqrt{9-25x^2}+C.
\end{aligned}$$

7.4.2 The General Approach

There are cases where a different trigonometric substitution is appropriate and useful. In fact the choices are mutually exclusive, and the form to be used can be deduced from the range of x-values in the domain of the original integrand, though one instead usually sees how problematic terms would simplify. In the chart below, we use x, though "x" can be x or $5x$, or similar, with the necessary algebra and calculus to compensate. Also for simplicity, we assume $a > 0$:[19]

Integral contains: Substitute:	$\sqrt{a^2-x^2}$ $x = a\sin\theta$	$\sqrt{a^2+x^2}$ $x = a\tan\theta$	$\sqrt{x^2-a^2}$ $x = a\sec\theta$
Motivation:	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{a^2+u^2} = a\sec\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
Range of x: Range of θ:	$-a \le x \le a$ $\theta \in [-\pi/2, \pi/2]$	$x \in \mathbb{R}$ $\theta \in (-\pi/2, \pi/2)$	$x \in (-\infty, -a] \cup [a, \infty)$ $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$
			

The graphs of the positions of θ are important for verification purposes, as well as for filling in with x-expressions the trigonometric functions of θ that usually arise from the trigonometric form of the antiderivative.

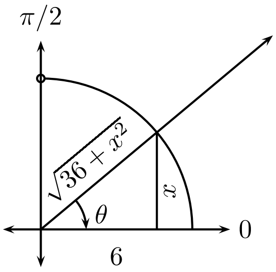
Example 7.4.4 *Compute* $\displaystyle\int \frac{1}{\sqrt{36+x^2}}\,dx$.

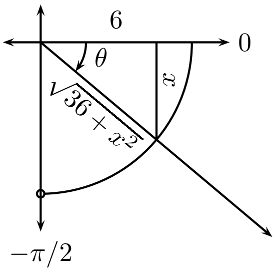
Solution: Here we will let $x = 6\tan\theta$. Note that as part of the method (but usually left unstated), $-\pi/2 < \theta < \pi/2 \iff -\infty < x < \infty$.

$$\int \frac{1}{\sqrt{36+x^2}}\,dx = \int \frac{1}{\sqrt{36+36\tan^2\theta}} \cdot 6\sec^2\theta\,d\theta = \int \frac{6\sec^2\theta}{\sqrt{36\sec^2\theta}}\,d\theta = \int \frac{6\sec^2\theta}{6\sec\theta}\,d\theta$$

$$\left.\begin{array}{r} x = 6\tan\theta \\ \implies dx = 6\sec^2\theta\,d\theta \end{array}\right\} = \int \sec\theta\,d\theta = \ln\lvert\sec\theta+\tan\theta\rvert + C_1.$$

At this point we have the integral solved in terms of θ. To get back to x, we draw the relevant triangles in which $x = \tan\theta$, i.e., $\tan\theta = x/6$. Note $\sec\theta = \left(\sqrt{36+x^2}\right)/6$ in both quadrants.





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[19] In fact, sometimes these substitutions are useful even when the radicals are not present, particularly for the tangent case, such as in computing $\int \frac{1}{(x^2+a^2)^n}\,dx$. So we can perhaps make a general statement that these substitutions should be considered if we have powers of (a^2-x^2), (a^2+x^2) or (a^2-x^2), respectively, when previous methods will not work, and the range of values for x and θ are compatible.

Thus

$$\begin{aligned}\int \frac{1}{\sqrt{36+x^2}}\,dx &= \ln|\sec\theta+\tan\theta|+C_1\\ &= \ln\left|\frac{\sqrt{36+x^2}}{6}+\frac{x}{6}\right|+C_1\\ &= \ln\left|\sqrt{36+x^2}+x\right|-\ln 6+C_1\\ &= \ln\left|\sqrt{36+x^2}+x\right|+C.\end{aligned}$$

It is customary to have the solution in simplest possible form, which is why we expanded the logarithm and absorbed the $-\ln 6$ term into the constant. (One usually only puts the subscript on the C_1 term after further lines reveal it is useful.)

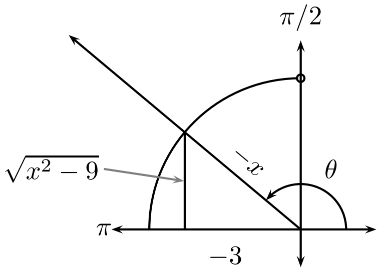
As we see in the drawings of the relevant angles, when we use a tangent-type subsitution we get the same, simple expressions for the derived sides regardless of which of the two quadrants holds the terminal ray of θ. It is not always the case with the secant substitutions.

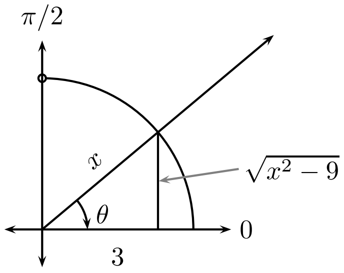
Example 7.4.5 *Compute* $\displaystyle\int\sqrt{x^2-9}\,dx$.

<u>Solution</u>: *Here we let $x=3\sec\theta$.*

$$\begin{aligned}\int\sqrt{x^2-9}\,dx &= \int\sqrt{9\sec^2\theta-9}\cdot 3\sec\theta\tan\theta\,d\theta\\ \left.\begin{array}{r}x=3\sec\theta\\ \implies\ dx=3\sec\theta\tan\theta\,d\theta\end{array}\right\} &= \int\sqrt{9\tan^2\theta}\cdot 3\sec\theta\tan\theta\,d\theta\\ &= \int 3|\tan\theta|\cdot 3\sec\theta\tan\theta\,d\theta\end{aligned}$$

Recall that our choices for θ terminate in either the first or second quadrant when we use a secant-type substitution, but while $\tan\theta\ge 0$ in Quadrant I, we have $\tan\theta\le 0$ in Quadrant II. These coincide with the cases x positive and x negative (or more precisely $x\ge 3$ and $x\le -3$), respectively. Below we graph the two cases, noting that $\sec\theta=x/3$, i.e., $\cos\theta=3/x$, but also that the "hypotenuse" must be positive in each case, as dictated by trigonometric theory.





The signs of the two "legs" of the representative triangles are also useful in checking the expression for the hyptoenuse. Note $\sec\theta=x/3=(-x)/(-3)$, and also that $-x>0$ for the case θ in QII.

For this particular example, the antiderivatives for the two cases differ by a factor of -1, so we do most of the work by finding the antiderivative for one case, and changing sign for the other. For simplicity we will compute the antiderivative for the case θ in Quadrant I first.

1. *Case* $x \geq 3$*:* $\displaystyle\int \sqrt{x^2-9}\,dx = 9\int \tan^2\theta\sec\theta\,d\theta$.

   This requires integration by parts, of the type where we "solve" for the integral. We will use

$$\begin{aligned} u &= \tan\theta & dv &= \sec\theta\tan\theta\,d\theta \\ du &= \sec^2\theta\,d\theta & v &= \sec\theta. \end{aligned}$$

$$\begin{aligned} (\mathcal{I}) &= uv - \int v\,du = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta \\ &= \sec\theta\tan\theta - \int(\tan^2\theta+1)\sec\theta\,d\theta = \sec\theta\tan\theta - \int\sec\theta\,d\theta - \int\tan^2\theta\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - \int\sec\theta\,d\theta - (\mathcal{I}). \end{aligned}$$

   Solving for $(\mathcal{I})$*, we have*

$$\begin{aligned} 2(\mathcal{I}) &= \sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C_1 \\ \implies (\mathcal{I}) &= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C. \end{aligned}$$

   Using our definition of $\sec\theta$ *and the previous diagram (for QI), we have*

$$\begin{aligned} (\mathcal{I}) &= \frac{1}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} - \frac{1}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right| + C_2 \\ &= \frac{x\sqrt{x^2-9}}{18} - \frac{1}{2}\left[\ln\left|x+\sqrt{x^2-9}\right| - \ln 3\right] + C_1 \\ &= \frac{x\sqrt{x^2-9}}{18} - \frac{1}{2}\ln\left|x+\sqrt{x^2-9}\right| + C. \end{aligned}$$

2. *Case* $x \leq -3$*: That is,* θ *in QII, where we will have the same antiderivative in* θ *except for a sign. Here* $\tan\theta \leq 0$*, so* $|\tan\theta| = -\tan\theta$*, so the trigonometric form of the antiderivative is the same as above except for an extra factor of* (-1)*:*

$$\begin{aligned} (\mathcal{I}) &= \int 3(-\tan\theta)\cdot 3\sec\theta\tan\theta\,d\theta \\ &= \frac{-1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{-1}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{-3} + \frac{1}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{-3}\right| + C_3 \\ &= \frac{x\sqrt{x^2-9}}{18} + \frac{1}{2}\left[\ln\left|x-\sqrt{x^2-9}\right| - \ln 3\right] + C_3 \\ &= \frac{x\sqrt{x^2-9}}{18} + \frac{1}{2}\ln\left|x-\sqrt{x^2-9}\right| + C. \end{aligned}$$

Summarizing,

$$\int\sqrt{x^2-9}\,dx = \begin{cases} \dfrac{1}{18}x\sqrt{x^2-9} - \dfrac{1}{2}\ln\left|x+\sqrt{x^2-9}\right| + C, & x \geq 3, \\ \dfrac{1}{18}x\sqrt{x^2-9} + \dfrac{1}{2}\ln\left|x-\sqrt{x^2-9}\right| + C, & x \leq -3. \end{cases}$$

Often, in a problem like this latest example we will know from the start what will be the range of interest of values of x. For instance, if we know $x \geq 0$ (more precisely, $x \geq 3$) we can finish the problem by drawing one diagram only. If this were a definite integral, for instance, we would know which range of x we need by looking at the endpoints of our integral.

It should be emphasized that only the secant-type trigonometric substitutions require us to check both quadrants, because $\sqrt{\sec^2\theta - 1} = \pm\tan\theta$ for the range of θ we use.[20]

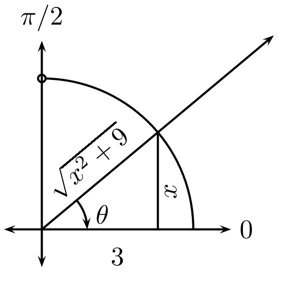
Sometimes, particularly for tangent substitutions, no radical is present in the original integral yet the trigonometric substitution is the method of choice.

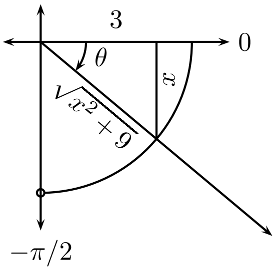
Example 7.4.6 *Compute* $\displaystyle\int \frac{1}{(x^2+9)^2}\,dx$.

Solution: Note that this is very different from the case where we have $(x^2+9)^1$ *in the denominator, which would eventually yield a arctangent, namely* $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$. *But it gives us a hint of what to do, namely let* $x = 3\tan\theta$.

$$\begin{aligned}
\int \frac{1}{(x^2+9)^2}\,dx &= \int \frac{1}{\left(9\tan^2\theta+9\right)^2}\cdot 3\sec^2\theta\,d\theta \\
\left.\begin{array}{r} x = 3\tan\theta \\ dx = 3\sec^2\theta\,d\theta \end{array}\right\} \quad &= \int \frac{3\sec^2\theta\,d\theta}{\left(9\sec^2\theta\right)^2} \\
&= \int \frac{3\sec^2\theta\,d\theta}{81\sec^4\theta} \\
&= \frac{1}{27}\int \cos^2\theta\,d\theta \\
&= \frac{1}{27}\cdot\frac{1}{2}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{27}\left[\frac{1}{2}\cdot\theta + \frac{1}{2}\sin 2\theta\right] + C \\
&= \frac{1}{54}\theta + \frac{1}{27}\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta + C.
\end{aligned}$$

Now the integral is solved in terms of θ, *so we need to get back to* x, *which we again do by looking at diagrams of the relation between* x *and* θ, *namely that* $\tan\theta = x/3$, *and* $\theta \in (-\pi/2, \pi/2)$.





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[20] There is an approach where one assumes $\theta \in [0, \pi 2/) \cup [\pi, 3\pi/2)$ or similar QI and QIII angles, and then $\sqrt{sec^2\theta - 1} = \tan\theta$. However we aviod this because sometimes the antiderivative includes the angle θ itself, which for our case would be $\theta = \sec^{-1}\frac{x}{3} = \cos^{-1}\frac{3}{x}$. This other approach simply redefines the arcsecant as well. The approach has many advantages (for instance the derivative of the arcsecant does not contain an absolute value), but at a cost of $\sec^{-1} z = \cos^{-1}\frac{1}{z}$.

Finishing off our integral is now fairly easy:

$$\begin{aligned}\int \frac{1}{\left(x^2+9\right)^2}\,dx &= \frac{1}{54}\theta + \frac{1}{27}\cdot\sin\theta\cos\theta + C\\ &= \frac{1}{54}\tan^{-1}x + \frac{1}{27}\cdot\frac{x}{\sqrt{x^2+9}}\cdot\frac{3}{\sqrt{x^2+9}} + C\\ &= \frac{1}{54}\tan^{-1}x + \frac{1}{9}\cdot\frac{x}{x^2+9} + C.\end{aligned}$$

While this is a powerful approach, it is not always the technique of choice. For example,

$$\int \frac{x}{\sqrt{x^2-9}}\,dx = \sqrt{x^2-9} + C,$$

from a simple substitution, or perhaps even an anticipation of the general form followed by a check of multiplicative constants. Similarly, if we replaced x with x^2 in the numerator we could integrate by parts. If we found ourselves integrating by parts twice, for instance (perhaps it is x^3 in the numerator) it may well be more efficient to use trigonometric substitution, or an integration by parts step may ultimately require it! But trigonometric substitution is usually not the method of choice if previous methods apply.

Exercises

1. Use trigonometric substitution to derive the general formula ($a > 0$)
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.$$
2. Use trigonometric substitution to derive the general formula ($a > 0$)
$$\int \frac{dx}{a^2+x^2} = \frac{x}{a}\tan^{-1}\frac{x}{a} + C.$$
3. Use trigonometric substitution to derive the general formula for $x > a > 0$:
$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{x}{a}\sec^{-1}\frac{x}{a} + C.$$
4. Compute $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$.
5. Compute $\displaystyle\int \left(9-x^2\right)^{3/2}\,dx$.
6. Compute $\displaystyle\int \frac{1}{(25+9x^2)^{5/2}}\,dx$.
7. Compute $\displaystyle\int \sqrt{x^2+2x}\,dx$. (Complete the square.)

7.5 Partial Fractions and Integration

In this section we are interested in techniques for computing integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx, \tag{7.40}$$

where $P(x)$ and $Q(x)$ are polynomials. This is not in general a simple problem, unless the integral in (7.40) is from very particular classes. However, with the techniques we explore here, we can break $\frac{P(x)}{Q(x)}$ into simpler fractions whose integrals are relatively easy. To see an advantage of such an approach, consider the following example.

Example 7.5.1 *Compute* $\displaystyle\int \frac{5x-1}{x^2-x-2}\,dx$.

<u>Solution</u>: Note that the numerator is not a simple (constant number) multiple of the derivative of the denominator, so substitution will not give a simple $\int \frac{1}{u}\,du$*-form.*

However, it happens that

$$\frac{5x-1}{x^2-x-2} = \frac{5x-1}{(x-2)(x+1)} = \frac{2}{x+1} + \frac{3}{x-2}. \tag{7.41}$$

Thus

$$\begin{aligned}\int \frac{5x-1}{x^2-x-2}\,dx &= \int \left(\frac{2}{x+1} + \frac{3}{x-2}\right) dx\\ &= 2\ln|x+1| + 3\ln|x-2| + C\\ &= \ln\left|(x+1)^2(x-2)^3\right| + C.\end{aligned}$$

In this section we will develop the methods needed to find expansions of fractions such as in (7.41). The idea is to reverse the high school algebra exercises, which would have us *combine* sums or differences of fractions into a single fraction. For purposes of integral calculus, it is almost always better to instead deal with several, simpler fractions than their combination into a single, more complicated fraction.

A significant amount of our work in this section will be algebraic, specifically, developing the method of decomposing a fraction $P(x)/Q(x)$ into simpler, "partial fractions." In order for the method to work, we will require $P(x)$ to have lower degree $Q(x)$. (If the degree of P is at least that of Q, we can use long division to write the function as a polynomial plus $r(x)/Q(x)$, where the degree of r is less than that of Q.)

Though not crucial for the calculus, we will spend the next subsection looking roughly at the theory behind the general form of these partial fraction decompositions (PFD's), in hopes it will help reinforce the rules themselves. In Subsection 7.5.2 we will definitively write the rules for PFD's without reference to integrals. Finally, we will see how to solve for the coefficients of a particular PFD, in the context of computing antiderivatives of these.

7.5.1 Theory Behind the Forms of PFD's (Optional)

The argument here is usually omitted from calculus texts, and instead left to linear algebra courses. However, the basic intuition is not difficult so we include it here, though the real work is in later subsections. In all of these we are looking at functions

$$\frac{P(x)}{Q(x)}, \qquad P \text{ and } Q \text{ polynomials, degree } P < \text{degree } Q. \tag{7.42}$$

Before stating the general rules for PFD's, we look at several examples illustrating the underlying theory.

Example 7.5.2 *For this example, we will argue in steps.*

1. *Consider all functions of the form*
$$\frac{ax+b}{(x+1)(x-2)}. \tag{7.43}$$

2. *Now there are two linearly independent*[21] *functions, specifically* $\frac{x}{(x+1)(x-2)}$ *and* $\frac{1}{(x+1)(x-2)}$ *which—with linear combinations—can give us any such function (7.43). Indeed,*
$$\frac{ax+b}{(x+1)(x-2)} = a\cdot\left[\frac{1}{(x+1)(x-2)}\right] + b\cdot\left[\frac{x}{(x+1)(x-2)}\right].$$
In a linear-algebraic sense, we would say the functions of the form (7.43) form a 2-dimensional space (or 2-dimensional vector space), because to specify such a function requires two constants, a *and* b.

3. *Now instead consider another 2-dimensional space of functions given by linear combinations of the form*
$$\frac{A}{x+1}+\frac{B}{x-2} = A\cdot\left[\frac{1}{x+1}\right] + B\cdot\left[\frac{1}{x-2}\right]. \tag{7.44}$$

4. *The functions* $\frac{1}{x+1}$ *and* $\frac{1}{x-2}$ *are indeed also linearly independent, so the set of all functions of the form (7.44) also forms a 2-dimensional vector space; to specify any such function requires specifying two constants,* A *and* B.

5. *Now notice that*
$$\frac{A}{x+1}+\frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)} = \frac{(A+B)x+(-2A+B)}{(x+1)(x-2)},$$
which is of form (7.43) with $a = A + B$ *and* $b = -2A + B$. *In other words, any function of the form (7.44) can also be written in the from (7.43).*

6. *This tells us that the two-dimensional space of functions* $\frac{A}{x+1}+\frac{B}{x-2}$ *is contained in the two-dimensional space of functions* $\frac{ax+b}{(x+1)(x-2)}$. *It is a fact of linear algebra that the only way for a two-dimensional space to be contained in another two-dimensional space is for them to be the same spaces. (Think about a plane being contained in another plane, and realize that they must then be the same plane.)*

7. *Finally, since (by 6 above) the space of all functions of the form* $\frac{ax+b}{(x+1)(x-2)}$ *is the* **same** *as the space of all functions of the form* $\frac{A}{x+1}+\frac{B}{x-2}$, *it follows that any function of the form* $\frac{ax+b}{(x+1)(x-2)}$ *can also be written in the form* $\frac{A}{x+1}+\frac{B}{x-2}$.

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[21] We call functions $f_1, f_2, \cdots, f_n$ *linearly independent* if and only if it is impossible to write any of these as linear combinations of the others. In other words, we *exclude* cases where there exist *constants* $a_1, \cdots, a_{k-1},\ a_{k+1}, \cdots, a_n \in \mathbb{R}$ such that
$$f_k = a_1f_1 + a_2f_2 + \cdots + a_{k-1}f_{k-1} + a_{k+1}f_{k+1} + \cdots + a_nf_n, \qquad \textit{i.e.},$$
$$(\forall x)[\qquad f_k(x) = a_1f_1(x) + a_2f_2(x) + \cdots + a_{k-1}f_{k-1}(x) + a_{k+1}f_{k+1}(x) + \cdots + a_nf_n(x) \qquad].$$

Notice that functions of the form (7.43) are indeed also of the form $P(x)/Q(x)$ where P is of degree less than Q, since the degree of P is at most 1 (zero if $a = 0$) and the degree of Q is 2.

The argument above guarantees that a PFD such as (7.41) exists. It is more desirable for integration purposes to have form (7.44) than (7.43).

Example 7.5.3 *An argument similar to that of the previous example shows that the following forms give exactly the same functions:*

$$\frac{ax^2+bx+c}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}. \tag{7.45}$$

Of course a, b, c are likely to differ from A, B, C. Here the underlying sets of linearly independent functions are, respectively,

$$U = \left\{ \frac{x^2}{(x+1)(x+2)(x+3)}, \frac{x}{(x+1)(x+2)(x+3)}, \frac{1}{(x+1)(x+2)(x+3)} \right\},$$
$$V = \left\{ \frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3} \right\}.$$

Both sets of vectors span[22] *3-dimensional spaces. It is not hard to see that functions on the right-hand side of (7.45) can also be in the form on the left. Indeed, if we combine the fractions on the right, we get*

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} = \frac{\overbrace{A(x+2)(x+3)}^{\text{degree } \le 2} + \overbrace{B(x+1)(x+3)}^{\text{degree } \le 2} + \overbrace{C(x+1)(x+2)}^{\text{degree } \le 2}}{(x+1)(x+2)(x+3)},$$

which gives us a polynomial in the numerator with degree at most 2, as on the left-hand side of (7.45). Here we have the span of V contained in the span of U, though they are both 3-dimensional spaces. Thus they must be the same spaces (we have one 3-dimensional space inside of another, so they must be the same!), so in fact, anything written like the left-hand side of (7.45) can be written like the right-hand side. (In a later subsection we will show how to find A, B, C given a, b, c.)

It should be clear that integrating a function written like the right-hand side of (7.45) is likely much simpler than integrating one in the form on the left.

Example 7.5.4 *Next we argue that the following forms describe the same (space of) functions:*

$$\frac{ax^2+bx+c}{(x+7)^3} = \frac{A}{x+7} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3}. \tag{7.46}$$

The underlying sets of linearly independent functions are, respectively, $\left\{ \frac{x^2}{(x+7)^3}, \frac{x}{(x+7)^3}, \frac{1}{(x+7)^3} \right\}$ *and* $\left\{ \frac{1}{x+7}, \frac{1}{(x+7)^2}, \frac{1}{(x+7)^3} \right\}$*, both spanning 3-dimensional spaces. To show they are the same spaces, we note that*

$$\frac{A}{x+7} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3} = \frac{A(x+7)^2 + B(x+7) + C}{(x+7)^3},$$

---

[22] The noun form of **span** has a precise technical meaning. The *span* of "vectors" $v_1, v_2, \cdots, v_n$ is the set of all possible linear combinations of those vectors. Thus for example

$$\text{Span}\left\{ \frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3} \right\} = \left\{ a \cdot \left[\frac{1}{x+1}\right] + b \cdot \left[\frac{1}{x+2}\right] + c \cdot \left[\frac{1}{x+3}\right] \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

We would then say that the functions (vectors, in the linear algebra sense) $\frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3}$, taken together, *span* the set described above.

which eventually simplifies to the form on the left-hand side of (7.46). Arguing as before, the underlying sets of linearly independent functions must span the same 3-dimensional spaces, so anything of the form on the left-hand side of (7.46) can be written also in the form on the right-hand side.

Example 7.5.5 *For our last example, we claim the following forms give the same functions:*

$$\frac{ax^4+bx^3+cx^2+dx+e}{x^3(x^2+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+1}. \tag{7.47}$$

Here the spanning sets of linearly independent functions are respectively

$$\begin{aligned} U&=\left\{\frac{x^4}{x^3(x^2+1)},\ \frac{x^3}{x^3(x^2+1)},\ \frac{x^2}{x^3(x^2+1)},\ \frac{x}{x^3(x^2+1)},\ \frac{1}{x^3(x^2+1)}\right\},\\ V&=\left\{\frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \frac{x}{x^2+1},\ \frac{1}{x^2+1}\right\}. \end{aligned}$$

Again, the form on the right of (7.47) can be rewritten as below, simplifying into the form on the left-hand side of (7.47) as

$$\frac{Ax^2(x^2+1)+Bx(x^2+1)+C(x^2+1)+(Dx+E)(x^2+1)}{x^3(x^2+1)}.$$

(Note that this has numerator of degree at most 4.) Because both sides of (7.47) must therefore describe exactly the same functions in a 5-dimensional vector space, it follows that anything written in the form on the left of (7.47) can also be written in the form on the right.

In the next subsection we generalize the logic of the examples above to write exact rules for the form of a PFD based upon the original fraction's denominator. Then in Subsection 7.5.3 we look at three methods of finding the coefficients, A, B, C, etc., of the PFD expansion, and immediately apply the methods to problems of computing integrals of such functions.

7.5.2 Partial Fraction Decompositions: The Rules

It is a fact of algebra (corollary to the Fundamental Theorem of Algebra) that any polynomial with real coefficients can be factored uniquely—up to rearrangement of multiplicative constants—into powers of linear terms $(ax+b)^n$ and powers of irreducible quadratic[23] terms $(ax^2+bx+c)^m$ with real coefficients. So for instance,

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1),$$

and there is no other way to factor it, except for instance $2(x-1)(\frac{1}{2}x^2+\frac{1}{2})$, etc. With that in mind, the rules for partial fraction decompositions follow.

First, we are given a rational function $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.

0. In 1 and 2 below we assume deg $P <$ deg Q. If the deg $P \geq$ deg Q, then first we apply poynomial long division to achieve

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)},$$

where p, r are polynomials and deg $r <$ deg Q. Then the following rules apply to $\frac{r(x)}{Q(x)}$.

1. If $(ax+b)^n$, where $a \neq 0$ occurs as a factor in $Q(x)$, then the partial fraction decomposition (PFD) of $\frac{P(x)}{Q(x)}$ will contain terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

2. If ax^2+bx+c is an irreducible quadratic, and $(ax^2+bx+c)^m$ occurs as a factor in $Q(x)$, then the PFD of $\frac{P(x)}{Q(x)}$ will contain terms

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

The application of these rules can be somewhat confusing at first, so we will look at several examples before proceeding to solve for the coefficients A_1, B_1, etc. For the second case, we will mostly be interested in irreducible quadratics of the form x^2+k^2, where $k > 0$. Note that we will usually use letters without subscripts, such as A, B, C, and so on for our PFD coefficients (to be found later).

Example 7.5.6 *Write the partial fraction decompositions for the given rational functions:*

- $\displaystyle \frac{3x^5-11x^3+15x-2}{(x+1)^2(x-3)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3} + \frac{F}{(x-3)^4}.$
- $\displaystyle \frac{1}{(x+1)^2(x-3)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3} + \frac{F}{(x-3)^4}.$

---

[23] It is easy to see when a quadratic term is "irreducible over the real numbers," meaning we cannot write it as $(ex+f)(gx+h)$, where $e, f, g, h \in \mathbb{R}$, the latter being equivalent to there being real numbers α, β such that the polynomial is zero there (i.e., at $\alpha = -f/e$, $\beta = -h/g$). Using the quadratic formula, it is plain that no such real solutions to the quadratic being zero occur if and only if $b^2 - 4ac < 0$ (i.e., when the term under the radical in the quadratic formula is negative).

In both cases, we had a polynomial of degree less than 6 divided by a polynomial of degree 6, so "Rule 0" is not invoked. We also had two factors of $x+1$ in the denominator, so we needed a constant over the first power, plus another constant over the second power, of $x+1$. With $(x-3)$ appearing to the third power in the denominator, we needed a constant over each of the first, second, and third powers of $x+3$. (Of course the choice of constants A,B,C,D,E will be different for these two functions above, but the form of their PFD's is the same.)

We do not want to be redundant in our PFD's, so if $Q(x)$ contains the factor $(x-3)^4$ but does not contain $(x-3)^5$, for instance, we require constants divided by $(x-3)$, $(x-3)^2$, $(x-3)^3$ and $(x-3)^4$ (but not $(x-3)^5$). Now one could say that such a $Q(x)$ also contains $(x-3)^2$, but we do not then require in our PFD constants divided by $(x-3)$ and $(x-3)^2$ *again*, since these are already taken care of by those required by the factor $(x-3)^4$ in $Q(x)$.

To rephrase the rules in light of the last paragraph, if exactly n factors of $(ax+b)$ appear in $Q(x)$, then the PFD contains terms $\frac{A_1}{ax+b}+\cdots+\frac{A_n}{(ax+b)^n}$. If exactly m factors of (ax^2+bx+c) appear, with $b^2-4ac<0$, then the PFD contains terms $\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$.

Example 7.5.7 *Here are more PFD expansion forms. (We do not solve for the coefficients yet.)*

- $\dfrac{x^4+x+1}{x^3(x^2+9)^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{Dx+E}{x^2+9}+\dfrac{Fx+G}{(x^2+9)^2}.$
- $\dfrac{1}{(x^2+1)(x^2+4)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+4}.$
- $\dfrac{x^5-8}{x(2x+1)^2(9-x)^3}=\dfrac{A}{x}+\dfrac{B}{2x+1}+\dfrac{C}{(2x+1)^2}+\dfrac{D}{9-x}+\dfrac{E}{(9-x)^2}+\dfrac{F}{(9-x)^3}.$
- $\dfrac{2}{x^2-5}=\dfrac{2}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)}=\dfrac{A}{x-\sqrt{5}}+\dfrac{B}{x+\sqrt{5}}.$
- $\dfrac{1}{x^4-1}=\dfrac{1}{(x^2-1)(x^2+1)}=\dfrac{1}{(x+1)(x-1)(x^2+1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{Cx+D}{x^2+1}.$

The last two PFD's above required us to factor the denominators before we started to implement the rules. Note that we must be careful to identify factors which are truly distinct. Consider the following:

$$\frac{1}{x(x-3)(3x-9)}=\frac{1}{3x(x-3)^2}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{(x-3)^2}.$$

The factors $(x-3)$ and $(3x-9)$ were not really distinct factors, but were constant multiples of eachother. If we do not notice this we will find ourselves attempting a PFD with $\frac{A}{x}+\frac{B}{x-3}+\frac{C}{3x-9}$, but the "$B$" and "$C$" terms are not independent, so we will miss one dimension of possibilities for our PFD. Note also that the factor $\frac{1}{3}$ can be included in the first PFD term (i.e., we could replace $\frac{A}{x}$ with $\frac{A}{3x}$, of course giving a different "A"), or its influence absorbed into the A, B and C terms. We will usually opt for the latter approach (as we did above).

7.5.3 Finding the Coefficients for PFD's

There are two main methods, and one auxiliary method, for finding the coefficients A, B, etc., for a PFD. The most efficient method for a particular PFD is usually a mixture of the two main methods; perhaps the first method can be used to find A and C, and the second to find B, for instance. Efficiently computing the coefficients is thus somewhat of an art.[24]

The methods are based upon some properties of polynomials. Consider two polynomials

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, \\ g(x) &= b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0. \end{aligned}$$

The statement that $f(x) = g(x)$ is an "equality of polynomials," i.e., that $f(x)$ and $g(x)$ are the *same* polynomial is equivalent to each of the following two conditions (separately):[25]

1. $(\forall x \in \mathbb{R})[f(x) = g(x)]$. In other words, f and g are the same functions.
2. $(\forall i \in \{1, 2, \cdots, \max\{m, n\}\})[a_i = b_i]$, that is, all the coefficients are the same. (Note that it is possible, for instance, that $m < n$, in which case we just take $b_{m+1}, \cdots, b_n = 0$.)

Furthermore, if $f(x)$ and $g(x)$ are the same polynomials, then $f'(x) = g'(x)$, $f''(x) = g''(x)$, $f'''(x) = g'''(x) \cdots$, in the sense of being the same polynomials, and so 1 and 2 from above apply to these derivatives as well.

Our first method will exploit 1, the second 2, and the auxiliary method will make use of final observation about derivatives. The first method essentially "probes" the two polynomials at different points, usually chosen strategically, to get some quick information out of a polynomial equality, and is often called an "evaluation method." The second method is often referred to as "comparing coefficients," and can also be useful for finding quick information. The auxiliary method exploits the fact that the first methods can be applied to the derivatives (of any order) of f and g to get further information quickly.

Example 7.5.8 *Compute the integral* $\displaystyle\int \frac{1}{x^2 - 5x + 6}\,dx$.

Solution: Here we have a degree-0 polynomial divided by a degree-2 polynomial, so the PFD rules apply. Now one usually writes the PFD form of the integrand, complete with the unknown coefficients, before proceeding to the methods of computing the coefficients. In other words, our first step would be to write:

$$\int \frac{1}{x^2 - 5x + 6}\,dx = \int \frac{1}{(x-2)(x-3)}\,dx = \int \left[\frac{A}{x-2} + \frac{B}{x-3}\right] dx.$$

The next two lines can be skipped with practice, though the first time one works this section they are worth writing so the mechanics of the method can be understood and reinforced. First we write the algebraic step (PFD) which was contained in the rewriting of the integrands above:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

This came from the fact that we have $x - 2$ as a factor in the denominator, but only once, and the same for $x - 3$. Next we multiply both sides by **the denominator on the left***:*

$$(x-2)(x-3)\left[\frac{1}{(x-2)(x-3)}\right] = (x-2)(x-3)\left[\frac{A}{x-2} + \frac{B}{x-3}\right].$$

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[24] In fact either method is—strictly speaking—sufficient, and indeed there are textbooks which teach only one or the other method. However, trying to fit a particularly complicated PFD into any single method will make for much more difficult computations than are necessary. That said, for computer programming one would likely choose one method and let the computer calculate the coefficients by "brute force."

[25] Some texts use the notation $f(x) \equiv g(x)$, read "$f(x)$ is identically equal to $g(x)$." In other words, $f(x)$ and $g(x)$ are the same functions. The word *identically* is in the spirit of, for instance, trigonometric identities, so one could write for example $\sin^2\theta + \cos^2\theta \equiv 1$. Of course we have a different use of the symbol "$\equiv$," and will thus refrain from using it in this context, but the reader should be aware of this common alternative use of the symbol.

On the left, the whole denominator cancels and we have the numerator of the original fraction. On the right we have to distribute the $(x-2)(x-3)$ *across the sum in the brackets. For the "A" term the* $(x-2)$ *cancels, while for the "B" term the* $(x-3)$ *cancels, giving us an equality of polynomials:*

$$1 = A(x-3) + B(x-2). \tag{7.48}$$

Because this is an equality of polynomials ($f(x) = g(x)$ *where* $f(x) = 1$ *and* $g(x) = A(x-3) + B(x-2)$*), it must be true for any* $x \in \mathbb{R}$*. Now we choose two values of* x *strategically.*

$$\begin{aligned} &\underline{x=3}: && 1 = A(3-3) + B(3-2) \implies \boxed{1 = B} \\ &\underline{x=2}: && 1 = A(2-3) + B(2-2) \\ & && \implies 1 = -A \implies \boxed{A = -1}. \end{aligned}$$

Now we summarize what we have so far, and compute the desired integral:

$$\begin{aligned} \int \frac{1}{x^2-5x+6} &= \int \left[\frac{-1}{x-2} + \frac{1}{x-3}\right] dx \\ &= -\ln|x-2| + \ln|x-3| + C \\ &= \ln\left|\frac{x-3}{x-2}\right| + C. \end{aligned}$$

(The last step is not necessary, but for reasons of style many textbooks combine logarithmic terms into a single logarithm.)

Because (7.48) was an equality of polynomials (meaning the polynomial on the left is *the same polynomial* as that on the right[26]), we could substitute any number for x in (7.48) and still have a true statement. Fortunately, there were choices which could eliminate an unknown, leaving an equation in the other unknown which is easily solved.

The second method for finding A and B (not preferred here but not terribly difficult here either) is to look at the coefficients of the polynomials on the left-hand side and right-hand side of (7.48), and realize that the coefficients of the various powers of x must agree for these to be the *same* polynomials. Though perhaps not necessary for this simple case, one sometimes expands the right-hand side and collects like terms

$$1 = (A+B)x + (-3A-2B).$$

From this or just reading from (7.48), we can in turn set equal the coefficients of the x^1 terms and the constant (some say x^0) terms to get the following system of two equations in two unknowns:

$$\left\{\begin{array}{rcrcr} 0 & = & A & + & B \\ 1 & = & -3A & - & 2B. \end{array}\right. \tag{7.49}$$

The first equation came from the fact that there is no x^1-term on the left-hand side of (7.49), or alternatively, the x^1-term is $0x^1$ on the left. To solve such a system one might add three

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[26] It should be pointed out that when we write a PFD, for instance

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3},$$

we mean that these are the same functions as well, so once we find A and B, the right-hand side would simplify to become the left-hand side. To find A and B we actually solve the *polynomial* equality (7.48) for A and B.

Note also the distinction between "equations" such as $2x-1=5$, which is true only for $x=3$, and "equalities" such as $(x+1)^2 = x^2+2x+1$, true for all x, meaning the function $(x+1)^2$ is the same as the function x^2+2x+1.

times the first equation to the second, to get $B = 1$, and use that information in the first to get $A = -1$, as before.

Whenever the denominator of our function $P(x)/Q(x)$ has a linear factor $(ax+b)$, evaluating the associated *polynomial* equality—such as (7.48)—at that x-value which makes this linear factor zero (namely $x = -b/a$) will quickly yield one of the coefficients, since all but one term in the polynomial equation will have $(ax + b)$ as a factor, and therefore vanish at $x = -b/a$. Thus this first method should always be employed to find that coefficient if the denominator Q has a linear factor. If the denominator has all linear terms to the first power, then this "evaluation" method will quickly yield all coefficients.

Example 7.5.9 *Compute* $\displaystyle\int \frac{2x^2 - 3x + 2}{x(x+5)(2x+1)}\,dx$.

Solution *It is important to notice that the numerator is degree 2, and the denominator degree 3, so the PFD rules do apply.*

$$\int \frac{2x^2-3x+2}{x(x+5)(2x+1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1}\right] dx.$$

Eventually we will cease writing the next two lines, but to be sure we will include them here so that the logic is clear:

$$\begin{aligned} \frac{2x^2-3x+2}{x(x+5)(2x+1)} &= \frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1} \\ \implies x(x+5)(2x+1)\left[\frac{2x^2-3x+2}{x(x+5)(2x+1)}\right] &= x(x+5)(2x+1)\left[\frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1}\right] \\ \implies \qquad 2x^2-3x+2 &= A(x+5)(2x+1) + Bx(2x+1) + Cx(x+5). \end{aligned}$$

Into this last line we can now enter values for x which will quickly yield the coefficients.

$$\begin{aligned} \underline{x=0}: &\qquad 2 = A(5)(1) \implies \boxed{A = \frac{2}{5}} \\ \underline{x=-5}: &\qquad 20+15+2 = B(-5)(-9) \\ &\qquad \implies 37 = 45B \implies \boxed{B = \frac{37}{45}}. \\ \underline{x=-\frac{1}{2}}: &\qquad 2\cdot\frac{1}{4} - 3\cdot\left(-\frac{1}{2}\right) + 2 = C\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+5\right) \\ &\qquad \implies \frac{1}{2} + \frac{3}{2} + 2 = C\left(-\frac{1}{2}\right)\left(\frac{9}{2}\right) \\ &\qquad \implies 4 = -\frac{9}{4}C \implies \boxed{C = -\frac{16}{9}}. \end{aligned}$$

Putting this together with our original integral, we get

$$\begin{aligned} \int \frac{2x^2-3x+2}{x(x+5)(2x+1)}\,dx &= \int \left[\frac{2/5}{x} + \frac{37/45}{x+5} + \frac{-16/9}{2x+1}\right] dx \\ &= \frac{2}{5}\ln|x| + \frac{37}{45}\ln|x+5| - \frac{16}{9}\cdot\frac{1}{2}\ln|2x+1| + C \\ &= \frac{2}{5}\ln|x| + \frac{37}{45}\ln|x+5| - \frac{8}{9}\ln|2x+1| + C. \end{aligned}$$

Next we look at an example where all factors of $Q(x)$ are linear, but one of these linear factors appears to the second power.

Example 7.5.10 *Compute* $\displaystyle\int \frac{x+1}{x^2(x-5)(x+4)}\,dx$.

Solution: This time we will describe but omit the explicit multiplication step in the PFD.[27]

$$\int \frac{x+1}{x^2(x-5)(x+4)}\,dx = \int \left[\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-5}+\frac{D}{x+4}\right]dx,$$

where

$$\frac{x+1}{x^2(x-5)(x+4)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-5}+\frac{D}{x+4}.$$

Multiplying by $x^2(x-5)(x+4)$ then gives us

$$x+1 = Ax(x-5)(x+4)+B(x-5)(x+4)+Cx^2(x+4)+Dx^2(x-5). \tag{7.50}$$

With this equation, choosing $x = 0, 5, -4$ will yield three of the four coefficients.

$$\begin{aligned}
&\underline{x=0}: && 1 = B(-5)(4) \implies \boxed{B=-\frac{1}{20}}\\
&\underline{x=5}: && 6 = C(5^2)(9)\\
& && \implies \frac{2\cdot 3}{5^2\cdot 3\cdot 3} = C \implies \boxed{C=\frac{2}{75}}\\
&\underline{x=-4}: && -3 = D((-4)^2)(-9)\\
& && \implies -3 = D[-16\cdot 9]\\
& && \implies 3 = D\cdot 16\cdot 3\cdot 3 \implies \boxed{D=\frac{1}{48}}
\end{aligned}$$

This exhausts the evaluations which give equations in one coefficient. Next we have several methods for finding A.

Method 1. *Compare coefficients. In particular, we look at the highest-order x-terms which appear—at least initially—in the polynomial equality, which for (7.50) means the x^3-terms. Here we have no x^3-terms on the left, and on the right, even without a complete expansion, we can see that the x^3-terms will be $A+C+D$. (The middle-order terms are more difficult to read from (7.50).) Fortunately we already know the values of C and D, so we have enough information to find A:*

$$\begin{aligned}
&\underline{x^3\text{-term}}: && 0 = A+C+D\\
& && \implies 0 = A+\frac{2}{75}+\frac{1}{48} = A+\frac{2}{3\cdot 5^2}+\frac{1}{2^4\cdot 3}\\
& && \iff 0 = A+\frac{2\cdot 2^4+1\cdot 5^2}{3\cdot 5^2\cdot 2^4} = A+\frac{32+25}{3\cdot 5^2\cdot 2^4}\\
& && \iff -\frac{57}{1200} = A \implies \boxed{A=-\frac{19}{400}}
\end{aligned}$$

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[27] The pattern of cancellation, when we multiply the PFD by the original denominator $Q(x)$, should become second nature with a small amount of practice. That said, it is important to remember what we are doing (multiplying by $Q(x)$) to get from the PFD to the polynomial equality, and how the various factors cancel (or do not cancel) in that multiplication.

From this we can complete the integration:

$$\int \frac{x+1}{x^2(x-5)(x+4)}\,dx = \int \left[\frac{-\frac{19}{400}}{x} + \frac{-\frac{1}{20}}{x^2} + \frac{\frac{2}{75}}{x-5} + \frac{\frac{1}{48}}{x+4}\right] dx$$
$$= -\frac{19}{400}\ln|x| - \frac{1}{20}\cdot\frac{-1}{x} + \frac{2}{75}\ln|x-5| + \frac{1}{48}\ln|x+4| + C.$$

Method 2. *One can instead evaluate the polynomial equality (7.50) at still another x-value, though no such value will produce A alone:*

$$\underline{x=1}: \qquad 2 = A(1)(-4)(5) + B(-4)(5) + C(1)^2(5) + D(1)^2(-4).$$

Since we already know B, C and D, we can insert that information and solve for A.

Method 3. *This will be more useful later, but this method (referred to earlier as the* auxiliary *method) certainly applies. The idea is that we apply* $\frac{d}{dx}$ *to both sides of (7.50), which is valid because the left-hand side and right-hand side of (7.50) are the same* functions.

In order to use this method, it is useful to recall the generalized product rule. For three functions u(x), v(x) and w(x), for instance, we have

$$(uvw)' = u'vw + uv'w + uvw'.$$

For reference we recall (7.50), from which we then compute the derivatives. Equation (7.50) reads:

$$x+1 = Ax(x-5)(x+4) + B(x-5)(x+4) + Cx^2(x+4) + Dx^2(x-5).$$

$$\begin{aligned}\underline{\frac{d}{dx}}: \qquad 1 = {} & A[(1)(x-5)(x+4) + x(1)(x+4) + x(x-5)(1)] \\ & + B[(1)(x+4) + (x-5)(1)] \\ & + C[(2x)(x+4) + x^2(1)] \\ & + D[(2x)(x-5) + x^2(1)].\end{aligned}$$

Now when we evaluate this at x = 0, we see that we get

$$1 = A[(1)(-5)(4)] + B[(1)(4) + (-5)(1)] + 0 + 0.$$

This gives us $1 = -20A - B$, *so then* $A = (1+B)/(-20) = (19/20)/(-20) = -19/400$, *as before.*

A simple principle buried in the third method is the following:

Theorem 7.5.1 *If* $(x-a)^m$*, where* $m > 1$ *is a factor of a polynomial* $f(x)$*, then* $(x-a)^{m-1}$ *is a factor of* $f'(x)$*.*

For a proof, we note that $(x-a)^m$ being a factor of $f(x)$ is equivalent to $f(x) = (x-a)^m g(x)$, where $g(x)$ is another polynomial. Thus

$$f'(x) = (x-a)^m g'(x) + m(x-a)^{m-1}(1)g(x) = (x-a)^{m-1}\underbrace{[(x-a)g'(x) + mg(x)]}_{\text{polynomial}},$$

so indeed $(x-a)^{m-1}$ is a factor of $f'(x)$.

Now evaluating both sides of (7.50) at $x=0$ caused those terms with x and x^2 factors to vanish, leaving an equation with B only. When we differentiate (7.50), those terms with x^2 factors *still vanish*—because one power of x remains—leaving only the B-term (as before) and the A-term (which had a factor x but not x^2). Already knowing B, we could solve for A.

A few guidelines for efficiently finding the PFD coefficients should be made at this point.

1. When linear factors are present in $Q(x)$, it is best to exhaust this method for finding some of the coefficients easily. This means evaluating the relevant polynomial equality at each value for which $Q(x)=0$.
2. When those values are exhausted, we should next compare coefficients of the powers of x, particularly the highest power which occurs on the right-hand side.
3. If $(ax+b)^m$ is a factor of $Q(x)$, where $m>1$, and the first two methods fail to get all coefficients, then differentiation of the polynomial equality may yield more coefficients.
4. If there are still coefficients to be found, then further evaluations, differentiations, or coefficient comparisons should be implemented.

Example 7.5.11 *Compute* $\displaystyle\int \frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}\,dx.$

Solution: As usual we start with the PFD.

$$\frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}$$

$$5x^3-17x^2+19x-13=A(x-2)^3+B(x+1)(x-2)^2+C(x+1)(x-2)+D(x+1) \quad (7.51)$$

$$\begin{aligned}
\underline{x=-1}: &\qquad -5-17-19-13=A(-27)\\
&\qquad \implies -54=-27A \implies \boxed{A=2}\\
\underline{x=2}: &\qquad 5(8)-17(4)+19(2)-13=D(3)\\
&\qquad \implies -3=3D \implies \boxed{D=-1}\\
\underline{x^3\text{-term}}: &\qquad 5=A+B\\
&\qquad \implies 5=2+B \implies \boxed{B=3}.
\end{aligned}$$

While we could perform another evaluation ($x=0$ comes to mind), or look at another coefficient (prone to error), instead we will differentiate (7.51):

$$\begin{aligned}
15x^2-34x+19=&\,A[3(x-2)^2(1)]\\
&+B[(1)(x-2)^2+(x+1)\cdot 2(x-2)(1)]\\
&+C[(1)(x-2)+(x+1)(1)]\\
&+D(1).
\end{aligned}$$

Now we evaluated at $x=2$:

$$\begin{aligned}
15(4)-34(2)+19&=C(3)+D\\
\implies 11&=3C-1 \implies \boxed{C=4}.
\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}\,dx &= \int\left[\frac{2}{x+1}+\frac{3}{x-2}+\frac{4}{(x-2)^2}-\frac{1}{(x-2)^3}\right]dx\\ &= 2\ln|x+1|+3\ln|x-2|-\frac{4}{x-2}-\frac{1}{-2}\cdot\frac{1}{(x-2)^2}+C\\ &= \ln\left|(x+1)^2(x-2)^3\right|-\frac{4}{x-2}+\frac{1}{2(x-2)^2}+C.\end{aligned}$$

A quick corollary to our Theorem 7.5.1 is that if $(x-a)^m$ is a factor of a polynomial $f(x)$, then for $k<m$ we have $(x-a)^{m-k}$ is a factor of $f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$. This follows from repeated applications of the theorem, which can be paraphrased as saying that we lose at most one factor of $(x-a)$ for each derivative we take, until we run out of factors of $(x-a)$. If our latest example had $(x-2)^4$ in the denominator, we could have taken a second derivative of the corresponding polynomial equation, and then those terms with $(x-2)^3$ or $(x-2)^4$ will stil be zero at $x=2$, but the other terms would likely be nonzero.[28]

Now we turn our attention to PFD's where the denominators contain irreducible quadratic factors.[29] One problem with such factors is that they are nonzero for any $x\in\mathbb{R}$,[30] so the evaluation method's usefulness is limited in these cases. For such PFD's, we will need to rely more upon the coefficient comparison method to find our coefficients.

Example 7.5.12 *Compute* $\int \frac{4x^3-7x^2+31x-38}{x^4+13x^2+36}\,dx.$

Solution: PFD rules apply since the degree of the numerator is less than that of the denominator. We need to begin by factoring the denominator of the integrand, after which we can write the general form of the PFD.

$$\int\frac{4x^3-7x^2+31x-38}{x^4+13x^2+36}\,dx=\int\frac{4x^3-7x^2+31x-38}{(x^2+4)(x^2+9)}\,dx=\int\left[\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+9}\right]dx.$$

Now taking the second equation, we underlying PFD becomes the polynomial equality

$$4x^3-7x^2+31x-38=(Ax+B)(x^2+9)+(Cx+D)(x^2+4). \tag{7.52}$$

Now we look at the coefficients.[31]

$$\begin{aligned}\underline{x^3\text{-term}}:&\quad 4=A+C\\ \underline{x^2\text{-term}}:&\quad -7=B+D\\ \underline{x^1\text{-term}}:&\quad 31=9A+4C\\ \underline{x^0\text{-term}}:&\quad -38=9B+4D.\end{aligned}$$

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[28] Note that it is quite possible that $x-a$ is not a factor of a polynomial $f(x)$, but *is* a factor of $f'(x)$. That is the case when $x=a$ is a *critical point* of $f(x)$. For example, $x-1$ is not a factor of $f(x)=x^2-2x+21$, but is a factor of $f'(x)=2x-2=2(x-1)$.

Note also that the theorem applies to any linear factor $ax+b$, where $a\neq 0$, since $ax+b=a\left(x+\frac{b}{a}\right)$. Thus if x^m is a factor of a polynomial $f(x)$, the x^{m-1} is a factor of $f'(x)$, etc., as is the case if we replace x^m with $(ax+b)^m=a^m\left(x+\frac{b}{a}\right)$.

[29] Until the next section, we will not be able to integrate the general case where we have an irreducible quadratic factor to a power greater than 1, with some exceptional cases.

[30] Recall that if $f(x)$ is a polynomial of degree ≥ 1, then $f(a)=0\iff(x-a)$ is a factor of $f(x)$.

[31] Note that the constant ("x^0") term equation is what we would get if we evaluated (7.52) at $x=0$. It is easy to see that this is always the case.

Though this looks like (and is) four equations in four unknowns, in fact it "decouples" into two systems, each with two unknowns, since the first and third equations have only A and C, and the second and fourth have B and D only. We solve these in turn.

$$\begin{array}{rclcl} 4 &=& A &+& C \\ 31 &=& 9A &+& 4C \end{array} \qquad\qquad \begin{array}{rclcl} -7 &=& B &+& D \\ -38 &=& 9B &+& 4D \end{array}$$

For the first system, we multiply the first equation by -9 *and add to the second, to get* $-5 = 0A-5C \implies C=1$. *From that we have the original first equation giving* $A = 4-C = 4-1 = 3$.

For the second system, we do the same, that is, mutliply the first equation by -9 *and add to the second, giving* $63-38 = -5D \implies 25 = -5D \implies -5 = D$. *From the original first equation in that system, we then get* $B = -7-D = -7+5 = -2$.

Now we compute the integral, noting that it is easier if we break the PFD into four distinct terms:

$$\begin{aligned} \int \frac{4x^3-7x^2+31x-38}{(x^2+4)(x^2+9)}\,dx &= \int \left[\frac{3x}{x^2+4} - \frac{2}{x^2+4} + \frac{x}{x^2+9} - \frac{5}{x^2+9}\right] dx \\ &= \frac{3}{2}\ln(x^2+4) - \frac{2}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(x^2+9) - \frac{5}{3}\tan^{-1}\frac{x}{3} + C \\ &= \ln\sqrt{(x^2+4)^3(x^2+9)} - \tan^{-1}\frac{x}{2} - \frac{5}{3}\tan^{-1}\frac{x}{3} + C. \end{aligned}$$

In the example above, we used the following common integration formula, which is particularly useful in problems encountered in this section. It is derivable with the usual substitution methods, and not too difficult to verify by differentiation. The formula is the following:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C. \tag{7.53}$$

We also used

$$\int \frac{x}{x^2+k^2}\,dx = \frac{1}{2}\ln(x^2+k^2) + C,$$

assuming $k \neq 0$. Note that we do not need absolute values inside the logarithm since $x^2+k^2 \geq k^2 > 0$.

When we have irreducible quadratic factors in the denominator $Q(x)$, it is likely that we will need to compare coefficients.[32] Afterall, there are no real numbers which will make all but one of those coefficients vanish. (We can make two vanish with $x = 0$, but that still leaves two.) If linear terms are also present, however, the evaluation method will yield one or more of the coefficients quickly.

Example 7.5.13 *Compute* $\displaystyle\int \frac{12x^4+190x^2+13x-6}{(2x-1)(x^2+16)}\,dx.$

<u>Solution</u>: First we note that the numerator has degree which is not less than the denominator, so we must use long division. To do so we need to expand the denominator: $(2x-1)(x^2+16) = 2x^3-x^2+32x-16$.

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[32] Or something equivalent to comparing coefficients. For instance, $x = 0$ gives $-38 = 9B+4D$, and one derivative of (7.52) gives

$$12x^2 - 14x + 31 = (A)(x^2+9) + (Ax+B)(2x) + (C)(x^2+4) + (Cx+D)(2x),$$

which, when we consider the datum $x = 0$ gives $31 = 9A+4C$. Both of these we had before. More derivatives, evaluated at $x = 0$, give multiples of the other two equations in our system (four equations in four unknowns).

Now through polynomial long division we get

$$\frac{12x^4+190x^2+13x-6}{(2x-1)(x^2+16)}=\frac{12x^4+190x^2+13x-6}{2x^3-x^2+32x-16}=6x+3+\frac{x^2+13x+42}{2x^3-x^2+32x-16}. \tag{7.54}$$

Refactoring our denominator, our integral now becomes

$$\int\left[6x+3+\frac{x^2+13x+42}{(2x-1)(x^2+16)}\right]\,dx=\int\left[6x+3+\frac{A}{2x-1}+\frac{Bx+C}{x^2+16}\right]\,dx.$$

The first two terms are easy enough. For our PFD, we need only concern ourselves with the remaining fraction:

$$\frac{x^2+13x+42}{(2x-1)(x^2+16)}=\frac{A}{2x-1}+\frac{Bx+C}{x^2+16}.$$

The corresponding polynomial equation is then

$$x^2+13x+42=A(x^2+16)+(Bx+C)(2x-1). \tag{7.55}$$

We begin with an evalutation, followed by a coefficient comparison.

$$\begin{aligned}
\underline{x=\tfrac{1}{2}}:\qquad & \frac{1}{4}+\frac{13}{2}+42=A\left(\frac{1}{4}+16\right)\\
\implies & \frac{1+26+168}{4}=\frac{65}{4}A\\
& \implies 195=65A\implies\boxed{A=3}\\
\underline{x^2\text{-term}}:\qquad & 1=A+2B\\
& \implies 1=3+2B\implies\boxed{B=-1}.
\end{aligned}$$

Perhaps the simplest next step is to find C by evaluation of (7.55) at, say, $x=0$:

$$\begin{aligned}
\underline{x=0}:\quad & 42=16A-C\\
\implies & 42=16(3)-C\\
\implies & C=16(3)-33=48-16\implies\boxed{C=6}.
\end{aligned}$$

Thus our original integral, including the polynomial terms, becomes

$$\begin{aligned}
\int\frac{12x^4+190x^2+13x-6}{(2x-1)(x^2+16)}\,dx&=\int\left[6x+3+\frac{3}{2x-1}-\frac{x}{x^2+16}+\frac{6}{x^2+16}\right]\,dx\\
&=3x^2+3x+\frac{3}{2}\ln|2x-1|-\frac{1}{2}\ln(x^2+16)+\frac{6}{4}\tan^{-1}\frac{x}{4}+C\\
&=3x(x+1)+3\ln\sqrt{|2x-1|}-\ln\sqrt{x^2+16}+\frac{3}{2}\tan^{-1}\frac{x}{4}+C.
\end{aligned}$$

The second from the last line was complete; the last line just gives some alternative styles for the particular terms.

Of course with any new technique, we have to be sure that we do not neglect the earlier methods.

7.6 Miscellaneous Methods

In this section we will use completing the square, and other methods to rewrite several types of integrals into forms where we can more easily use either partial fractions or trigonometric substitution. We will also look at examples where a substitution will bring us to such forms. Finally, we will consider the use of integration tables, which can be found in numerous publications, but which require some sophistication to be used properly.