# Chapter 10

# Series of Constants

In this chapter we consider "infinite sums," which we call *series,* such as

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1}_{n=1} + \underbrace{a_2}_{n=1} + \underbrace{a_3}_{n=3} + \cdots . \tag{10.1}$$

The "sum" above begins with $a_1$, but we will often begin with a term $a_0$, or $a_2$, etc. It is not the beginning terms which determine if we can in fact compute such a sum, but rather it is the infinite "tail" of the series. This is reasonable because we can always, in principle, add as many terms together as we like, so long as there are finitely many of them. As with other calculus concepts, the tool which breaks the finite/infinite barrier is limit. Indeed, to make sense of a sum such as (10.1), we consider the $N$th *partial sum*,

$$S_N = \sum_{n=1}^{n} a_n = a_1 + a_2 + \cdots + a_{N-1} + a_N, \tag{10.2}$$

and then look at the *sequence* of these partial sums $S_1, S_2, S_3, \cdots$, i.e.,

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ S_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned}$$

and so on. To determine if (10.1) makes sense is then considered (by definition) to be equivalent to determining the behavior of the sequence $\{S_N\}_{N=1}^{\infty}$. We say the series (10.1) *converges* to $S \in \mathbb{R}$ if and only if $S_N \longrightarrow S$ as $N \to \infty$.

In a few cases we will actually be able to compute a simple formula for $S_N$, and thus be able to compute the series by taking $N \to \infty$. However, in many cases we cannot find a compact formula for $S_N$. In those cases we have to develop other methods for determining if the series converges at all, and if so, how to approximate the value of the series with as much precision as we require, by determining how large we require $N$ to be so that we can approximate the full series by $S_N$.

In this text we will take more steps than most other texts in developing the theory of series, since this topic is the source of much confusion for students. Indeed we devote this entire chapter to the topic of series of constant terms, leaving nonconstant terms for their own chapter. The concepts are intuitive—perhaps deceptively so—but require practice so that, for example, one can recognize when and where to apply a particular test of convergence or divergence.

In the next chapter we will look at functions defined by series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+\cdots. \qquad (10.3)$$

In fact most functions we have dealt with in this text can be written in the form above, at least on some open intervals, so such functions are very important theoretically. However there are very important functions which can be written in the form (10.3) but not by using the functions from our usual library (powers, logs, exponential, trigonometric and arctrigonometric functions). We deal extensively with functions of the form (10.3), also known as *power series,* in the next chapter.

## 10.1 Series and Partial Sums

As mentioned in the introduction to this chapter, the convergence of a series is defined as equivalent to the convergence of its partial sums. For convenience, we will define the $N$th partial sum to be the sum of all terms of the underlying sequence up to the term whose subscript is $N$. Thus if the sequence is series is $\sum_{n=k}^{\infty} a_n$, with underlying sequence $\{a_n\}_{n=k}^{\infty}$, then

$$S_N = \sum_{n=k}^{N} a_n = a_k + a_{k+1} + \cdots + a_N. \tag{10.4}$$

So for a series $a_0 + a_1 + a_2 + \cdots$, the partial sum $S_N = a_0 + a_1 + \cdots + a_N$ would actually have $N + 1$ terms, though we will still call it the $N$th partial sum. (Of course if $N < k$ we do not define an $N$th partial sum.)

**Example 10.1.1** *Consider the series* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$, *and find the first five partial sums.*

*Solution: We do this directly:*

$$\begin{aligned}
S_1 &= \sum_{n=1}^{1} \frac{(-1)^n}{n^2+1} = \frac{(-1)^1}{1^2+1} &&= \frac{-1}{2}\\
S_2 &= \sum_{n=1}^{2} \frac{(-1)^n}{n^2+1} = \frac{(-1)^1}{1^2+1} + \frac{(-1)^2}{2^2+1} &&= \frac{-1}{2} + \frac{1}{5} = -\frac{3}{10} = 0.3\\
S_3 &= \sum_{n=1}^{3} \frac{(-1)^n}{n^2+1} = \frac{(-1)^1}{1^2+1} + \frac{(-1)^2}{2^2+1} + \frac{(-1)^3}{3^2+1} &&= -\frac{3}{10} + \frac{-1}{10} = \frac{2}{10} = 0.2\\
S_4 &= \sum_{n=1}^{4} \frac{(-1)^n}{n^2+1} = \frac{(-1)^1}{1^2+1} + \frac{(-1)^2}{2^2+1} + \frac{(-1)^3}{3^2+1} + \frac{(-1)^4}{4^2+1} &&= S_3 + \frac{1}{17} = \frac{44}{170} \approx 0.2588235294\\
S_5 &= \sum_{n=1}^{5} \frac{(-1)^n}{n^2+1} = S_4 + \frac{(-1)^5}{5^2+1} &&= \frac{44}{170} + \frac{-1}{26} \approx 0.220361991.
\end{aligned}$$

Note we used the simple recursion relationship for partial sums of a series: given a series $\displaystyle\sum_{n=k}^{\infty} a_n$, and $N \geq k$ we have

$$S_{N+1} = S_N + a_{N+1}, \tag{10.5}$$

that is,

$$\begin{aligned}
S_{N+1} &= \sum_{n=k}^{N+1} a_n = \underbrace{a_k + a_{k+1} + \cdots + a_N}_{S_N} + a_{N+1} = \sum_{n=k}^{N} a_n + a_{N+1}\\
&= S_N + a_{N+1}, \text{ q.e.d.}
\end{aligned}$$

In a later section we will see that the series in the above example does in fact converge, though we can only approximate its exact value by computing $S_N$ for large values of $N$.

### 10.1.1 Telescoping Series

Telescoping series do occur on occasion, but the main reason they are included in most calculus textbooks is that their partial sums simplify in nice ways, leaving us able to compute their limits and thus the whole series. Indeed, the behavior of telescoping series is unusually "nice"—rivaled only by that of the much more important geometric series we will see later in this section—and therefore well-suited for early examples of the general notion of series.

The simplest type of telescoping series is one in which the terms added are themselves sums of two terms, constructed in such a way that there is cancellation such as the following:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} [b_n - b_{n-1}] \qquad (10.6)\\ &= (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) + \cdots .\end{aligned}$$

After a careful examination of the terms which appear in (10.6), it seems that all cancel except for $-b_0$. However we must be even more careful since there are infinitely many terms we are claiming we can cancel. The correct approach is to carefully examine the partial sums:

$$\begin{aligned}S_1 &= b_1 - b_0,\\ S_2 &= \cancel{b}_1 - b_0 + b_2 - \cancel{b}_1 = b_2 - b_0,\\ S_3 &= \cancel{b}_1 - b_0 + \cancel{b}_2 - \cancel{b}_1 + b_3 - \cancel{b}_2 = b_3 - b_0,\\ S_4 &= \cancel{b}_1 - b_0 + \cancel{b}_2 - \cancel{b}_1 + \cancel{b}_3 - \cancel{b}_2 + b_4 - \cancel{b}_3 = b_4 - b_0,\end{aligned}$$

and so on, whereby we can conclude that, for this simplest type of example (10.6), we have

$$S_n = b_n - b_0. \qquad (10.7)$$

Now such a series will therefore converge if and only if $\{b_n\}_{n=1}^{\infty}$ converges. If $b_n \longrightarrow B \in \mathbb{R}$ as $n \to \infty$, then by (10.7) we have $S_n \longrightarrow B - b_0$, whence $\sum_{n=1}^{\infty} [b_n - b_{n-1}] = B - b_0$.

More complicated telescoping series also occur, though the basic idea is that the partial sums can be written in such a way that all but a few terms found in the partial sums eventually cancel, and where we can compute the limits of those terms which do not.[1] Rather than memorizing the sample telescoping forms (10.6) and (10.7), it is better to consider each example separately, writing out the terms of $S_N$ for enough values of $N$ that the pattern emerges.

**Example 10.1.2** *Consider the series* $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n+1} - \frac{1}{n}\right]$. *Compute the form of each partial sum $S_N$ (as a function of $N$), and the value of the series if it converges.*

<u>Solution</u>: *We will write out a few partial sums longhand, from which the pattern will emerge.*

[1] It is interesting to visualize why the term *telescoping* is used to describe such a series. One of the *Webster's* dictionaries defines the intransitive verb form of *telescope* as follows:

> *to slide together, or into something else, in the manner of the tubes of a jointed telescope.*

For another example, a "telescoping antenna" comes to mind. The reader should keep such images in mind as we consider so-called telescoping series.

*Indeed, all but two terms will cancel in each of the following.*

$$\begin{aligned}
S_1 &= \left[\frac{1}{2} - 1\right] && = \frac{1}{2} - 1,\\
S_2 &= \left[\frac{1}{2} - 1\right] + \left[\frac{1}{3} - \frac{1}{2}\right] && = \frac{1}{3} - 1,\\
S_3 &= \left[\frac{1}{2} - 1\right] + \left[\frac{1}{3} - \frac{1}{2}\right] + \left[\frac{1}{4} - \frac{1}{3}\right] && = \frac{1}{4} - 1,\\
S_4 &= \left[\frac{1}{2} - 1\right] + \left[\frac{1}{3} - \frac{1}{2}\right] + \left[\frac{1}{4} - \frac{1}{3}\right] + \left[\frac{1}{5} - \frac{1}{4}\right] && = \frac{1}{5} - 1.
\end{aligned}$$

*From this we do indeed see a pattern in which*

$$S_N = \frac{1}{N+1} - 1.$$

*Taking $N \to \infty$, we see $S_N = \frac{1}{N+1} - 1 \longrightarrow 0 - 1 = -1$, and so we conclude that the series converges to $-1$, i.e.,*

$$\sum_{n=1}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n}\right] = -1.$$

Sometimes we need to do a little more work to detect a telescoping series, and its formula for $S_N$. Note that the general term of the added sequence terms, namely $\left[\frac{1}{n+1} - \frac{1}{n}\right]$, in our series above looks like a partial fraction decomposition if the variable is $n$. For that reason, when the general term can be written in a PFD, the series may in fact be telescoping. This is the case with the following example.

**Example 10.1.3** *Consider the series* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. *Compute a general formula for the $N$th partial sum $S_N$, and compute its limit, if $S_N$ converges, thereby computing the series.*

<u>Solution</u>: *Note first that there is no $S_1$ here. That said, the technique which we will use for this is to first look at the partial fraction decomposition (PFD) for $\frac{1}{n^2-1}$. Of course we need the denominator factored, giving us the form*

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}.$$

*Multiplying by $(n+1)(n-1)$ in the second equation then gives us*

$$1 = A(n-1) + B(n+1).$$

*Now we use the usual methods for computing the coefficients $A$ and $B$:*

$$\begin{aligned}
\underline{n=1}: &\qquad 1 = B(2) \implies \boxed{B = \frac{1}{2}}\\
\underline{n=-1}: &\qquad 1 = A(-2) \implies \boxed{A = -\frac{1}{2}}.
\end{aligned}$$

*From this we can rewrite our series*

$$\sum_{n=2}^{\infty} \left[\frac{-1/2}{n+1} + \frac{1/2}{n-1}\right] = \sum_{n=2}^{\infty} \left[\frac{1}{2}\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right].$$

*There is no $S_1$, so we begin with $S_2$.*

$$\begin{aligned}
S_2 &= \frac{1}{2}\left(\frac{-1}{3}+1\right) &&= \frac{1}{2}\left(\frac{-1}{3}+1\right),\\
S_3 &= \frac{1}{2}\left(\frac{-1}{3}+1\right)+\frac{1}{2}\left(\frac{-1}{4}+\frac{1}{2}\right) &&= \frac{1}{2}\left(\frac{-1}{3}+1+\frac{-1}{4}+\frac{1}{2}\right),\\
S_4 &= \frac{1}{2}\left(\frac{-1}{3}+1\right)+\frac{1}{2}\left(\frac{-1}{4}+\frac{1}{2}\right)+\frac{1}{2}\left(\frac{-1}{5}+\frac{1}{3}\right) &&= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}\right),\\
S_5 &= S_4+\frac{1}{2}\left(\frac{-1}{6}+\frac{1}{4}\right) &&= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}\right),\\
S_6 &= S_5+\frac{1}{2}\left(\frac{-1}{7}+\frac{1}{5}\right) &&= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}\right),\\
S_7 &= S_6+\frac{1}{2}\left(\frac{-1}{8}+\frac{1}{6}\right) &&= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{7}-\frac{1}{8}\right)
\end{aligned}$$

*By this point a pattern has clearly emerged:*

$$S_N = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N}-\frac{1}{N+1}\right),$$

*and so* $S_N \longrightarrow \frac{1}{2}\left[1+\frac{1}{2}-0-0\right] = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$ *as* $N\to\infty$. *We can thus conclude that*

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \sum_{n=2}^{\infty}\left[\frac{1}{2}\left(\frac{-1}{n+1}+\frac{1}{n-1}\right)\right] = \frac{3}{4}.$$

**Example 10.1.4** *Find $S_N$ and discuss the convergence (or divergence) of the series*

$$\sum_{n=0}^{\infty}\left[\sqrt{n+1}-\sqrt{n}\right].$$

<u>Solution</u>:

$$\begin{aligned}
S_0 &= \left[\sqrt{1}-\sqrt{0}\right] &&= \sqrt{1}-\sqrt{0},\\
S_1 &= \left[\sqrt{1}-\sqrt{0}\right]+\left[\sqrt{2}-\sqrt{1}\right] &&= \sqrt{2}-\sqrt{0},\\
S_2 &= \left[\sqrt{1}-\sqrt{0}\right]+\left[\sqrt{2}-\sqrt{1}\right]+\left[\sqrt{3}-\sqrt{2}\right] &&= \sqrt{3}-\sqrt{0},\\
S_3 &= \left[\sqrt{1}-\sqrt{0}\right]+\left[\sqrt{2}-\sqrt{1}\right]+\left[\sqrt{3}-\sqrt{2}\right]+\left[\sqrt{4}-\sqrt{3}\right] &&= \sqrt{4}-\sqrt{0},
\end{aligned}$$

*and so on, so that*

$$S_N = \sqrt{N+1}-\sqrt{0} = \sqrt{N+1} \longrightarrow \infty \text{ as } N\to\infty.$$

*Thus the series diverges (to infinity, to be more descriptive).*

Note that we could simplify our earlier expressions for $S_N$, since for instance $\sqrt{0}=0$, $\sqrt{1}=1$ and $\sqrt{4}=2$, but to do so would more likely obscure the pattern of cancellation.

### 10.1.2 Geometric Series

The class of series considered here is arguably the most important we will encounter. Many important series analyses depend upon how a particular series compares to, or mimics the behavior of, an appropriately chosen geometric series. As with the telescoping series, the geometric series is one for which we can actually compute a general formula for $S_N$, from which we can tell if the series converges, and if so compute its sum.

What makes a series $\sum a_n$ geometric is that there exists a constant $r \in \mathbb{R} - \{0\}$ such that

$$(\forall n)\left[\frac{a_{n+1}}{a_n} = r\right]. \tag{10.8}$$

In other words, such a series can be defined recursively by $a_{n+1} = r \cdot a_n$. (Note that this is equivalent to $a_n = r \cdot a_{n-1}$, so long as $a_{n-1}$ is defined.) Put more colloquially, a geometric series is one in which we get the next term by multiplying the present term by the same constant each time. Examples of geometric series follow:

- $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \qquad (r = 1/2),$
- $\displaystyle\sum_{n=2}^{\infty}\frac{6}{5^n} = \frac{6}{25} + \frac{6}{125} + \frac{6}{625} + \frac{6}{3125} + \cdots \qquad (r = 1/5),$
- $\displaystyle\sum_{n=1}^{\infty}\frac{2(-1)^n}{3^n} = -\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \cdots \qquad (r = -1/3),$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{2n}} = \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \frac{1}{6561} + \cdots \qquad (r = 1/9).$

Note that this last series can be rewritten $\sum_{n=1}^{\infty}\frac{1}{9^n}$, or even $\sum_{n=0}^{\infty}\left[\frac{1}{9}\cdot\left(\frac{1}{9}\right)^n\right]$. In fact, unlike the telescoping series, every geometric series can be written in the same form, namely[2]

$$\sum_{n=0}^{\infty}\alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots, \tag{10.9}$$

where

$$\alpha \text{ is the } \textit{first term} \text{ of the series, and} \tag{10.10}$$

$$r \text{ is the } \textit{constant } \textbf{ratio},\ a_{n+1}/a_n. \tag{10.11}$$

In the examples above, the first terms are $\alpha = 1, 6/25, -2/3, 1/9$ respectively. Each of the series above can be rewritten in $\Sigma$-notation in the form (10.9), starting with $n = 0$. For instance, the third series above can be rewritten, using $\alpha = -2/3$ and $r = -1/3$, as

$$\sum_{n=1}^{\infty}\frac{2(-1)^n}{3^n} = -\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \cdots = \sum_{n=0}^{\infty}\frac{-2}{3}\left(\frac{-1}{3}\right)^n.$$

In fact, once we know a series is geometric (that is, that $a_{n+1} = r \cdot a_n$ for each $n$), all we need to do is to identify $\alpha$ and $r$, and we can write the series in the exact $\Sigma$-notation form (10.9).

[2] With geometric series, it is understood that "$r^0$" represents 1, even though technically this is only correct if $r > 0$. In each general setting in which we follow the convention that $r^0$ is defined to be 1 (regardless of the sign of $r$), we will remark on this point.

**Example 10.1.5** *Write the series $4 + \frac{2}{3} + \frac{1}{9} + \frac{1}{54} + \frac{1}{324} + \cdots$ in the form (10.9).*

<u>Solution</u>*: Though perhaps not immediately obvious, in fact each successive term is $\frac{1}{6}$ times its immediate predecessor. The first term is 4. We translate these two facts as $\alpha = 4$ and $r = \frac{1}{6}$, and so this series is the same as the series*

$$\sum_{n=0}^{\infty} 4 \cdot \left(\frac{1}{6}\right)^n.$$

As with telescoping series, a geometric series allows for a simple formula for $S_N$. To use the formula, however, we need to make two assumptions:

1. that the series is already written in the form $\displaystyle\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots$, and

2. that $r \neq 1$.

As we have seen, the first requirement is easy enough to accomplish: we need only identify $\alpha$ (the first term in the geometric series) and $r$. The second requirement is for technical reasons we will encounter momentarily. We do not loose much in assuming $r \neq 1$, since in the case $r = 1$ the series is simply $\alpha + \alpha + \alpha + \cdots$, which is clearly a divergent series if $\alpha \neq 0$, and trivial if $\alpha = 0$.[3] Now we state our theorem.

**Theorem 10.1.1** *For a geometric series $\displaystyle\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots$, assuming $r \neq 1$, we have*

$$S_N = \frac{\alpha\left(1 - r^{N+1}\right)}{1 - r}. \tag{10.12}$$

<u>**Proof**</u>**:** The usual method of proof of (10.12) is to exploit the geometric nature of the series in the following way:

$$\begin{array}{rrcccccccccccc}
 & S_N & = & \alpha & + & \alpha r & + & \alpha r^2 & + & \cdots & + & \alpha r^N \\
\Longrightarrow & r \cdot S_N & = & \alpha r & + & \alpha r^2 & + & \alpha r^3 & + & \cdots & + & \alpha r^{N+1} \\
\hline
\Longrightarrow & (1-r)S_N & = & \alpha & + & 0 & + & 0 & + & 0 & - & \alpha r^{N+1}
\end{array}$$

In the first line we wrote the definition of $S_N$. In the next line we multiplied that equation by $r$. In the third line, the second line is subtracted from the first. In doing so, the terms $\alpha r, \alpha r^2, \cdots, \alpha r^N$ cancel, leaving only $\alpha - \alpha r^{N+1}$ on the right-hand side. This gives us

$$(1 - r)S_N = \alpha\left(1 - r^{N+1}\right).$$

Since we are assuming $r \neq 1$, we can divide by $1 - r$ and get (10.12), as desired.

To utilize (10.12), one needs to know $\alpha$, $r$ and $N$. Note that $N$ is not the number of terms, but the highest power of $r$ which occurs. In fact there are $N + 1$ terms added to arrive at $S_N$, since the first is $\alpha r^0$.

[3] We will not generally consider the case $\alpha = 0$ because it is trivial, and because we cannot identify a unique $r$. Indeed, if $\alpha = 0$, then any geometric recursion is valid, but our original method of defining $r$, namely (10.8) on page 805, is undefined if $\alpha = 0$.

**Example 10.1.6** *Consider the series* $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. *Find the sum of the first 9 terms.*

*Solution: What we are seeking here is* $S_8=\frac{\alpha\left(1-r^{8+1}\right)}{1-r}$, *where* $\alpha=1$ *and* $r=\frac{1}{2}$. *Thus*

$$S_8=\frac{1\left[1-\left(\frac{1}{2}\right)^9\right]}{1-\frac{1}{2}}=\frac{1-\frac{1}{512}}{\frac{1}{2}}=\frac{1-\frac{1}{512}}{\frac{1}{2}}\cdot\frac{512}{512}=\frac{512-1}{256}=\frac{511}{256}=1.99609375$$

The formula (10.12) also works when $r<0$.

**Example 10.1.7** *Consider the series* $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$. *Find the sum of the first 9 terms.*

*Solution: Again we want* $S_8$, *but while* $\alpha=1$ *as before, here we have* $r=-1/2$.

$$S_8=\frac{1\left[1-\left(\frac{-1}{2}\right)^9\right]}{1-\left(-\frac{1}{2}\right)}=\frac{1-\frac{-1}{512}}{\frac{3}{2}}=\frac{1+\frac{1}{512}}{\frac{3}{2}}\cdot\frac{512}{512}=\frac{512+1}{256\cdot 3}=\frac{513}{256\cdot 3}=\frac{171}{256}=0.66796875.$$

**Example 10.1.8** *Suppose one deposits into an account (without interest) one penny ($0.01) on the first day of a month, then deposits two pennies ($0.02) the next day, four pennies the next, and so on, each day depositing twice what was deposited the day before. How much money is in the account after the first week (7 payments), second week, third week, and thirty-first day?*

*Solution: This is the same as asking for partial sums of the series* $0.01+0.02+0.04+0.08+\cdots$. *This is a geometric series (10.9) with* $\alpha=0.01$ *and* $r=2$. *Here we have to be careful about* $N$, *since after the first day* $N=0$, *after the second* $N=1$, *etc. Now we compute the total deposit after*

- *1 week, i.e., 7 days, we have* $N=6$ *and*

$$S_6=\frac{0.01\left[1-2^7\right]}{1-2}=\frac{0.01\left[1-2^7\right]}{-1}=0.01\left(2^7-1\right)=0.01(127)=1.27.$$

- *2 weeks, i.e., 14 days, we have* $N=13$ *and (continuing the pattern above)*

$$S_{13}=\frac{0.01\left[1-2^{14}\right]}{1-2}=0.01\left(2^{14}-1\right)=0.01(16383)=163.83.$$

- *3 weeks, i.e., 21 days, we have* $N=20$ *and*

$$S_{20}=\cdots=0.01\left(2^{21}-1\right)=0.01(2,097,151)=20,971.51$$

- *31 days, so we have* $N=30$, *and*

$$S_{30}=\cdots=0.01\left(2^{31}-1\right)=0.01(2,147,483,647)=21,474,836.47.$$

This latest example illustrates that, when $r>1$, the function $N\mapsto S_N$ is essentially exponential. Indeed, *as a function of* $N$,

$$S_N=\frac{\alpha}{1-r}\left[1-r^{N+1}\right]=\frac{\alpha}{1-r}+\frac{-\alpha\cdot r^{N+1}}{1-r}=\frac{\alpha}{1-r}+\left[\frac{\alpha}{r-1}\cdot r\right]\cdot r^N=A+Br^N,$$

where $A=\frac{\alpha}{r-1}$ and $B=\frac{\alpha r}{r-1}$. Thus as a function of $N$, $S_N$ is basically a vertical translation of an exponential growth $Br^N$, assuming again that $r>1$.[4] This partially explains why some use the term "geometric growth" when referring to exponential growth.

[4] If $r\in(0,1)$ we get a translation of exponential decay; if $r\in(-1,0)$ we get a kind of "damped oscillation"; if $r=-1$ we get steady oscillation; and if $r<-1$ we get a growing oscillation. Details are left to the reader.

### 10.1.3 Convergence/Divergence in Geometric Series

Now we look at necessary and sufficient conditions for a geometric series to converge. If a given geometric series does converge, we compute its sum. Our result is the following:

**Theorem 10.1.2** *For a geometric series* $\displaystyle\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots$, *where* $\alpha \neq 0$,

1. *the series* **converges** *if and only if* $|r| < 1$, *i.e.,* $r \in (-1, 1)$;
   *equivalently, the series* **diverges** *if and only if* $|r| \geq 1$, *i.e.,* $r \in (-\infty, -1] \cup [1, \infty)$.
2. *if* $|r| < 1$, *then the series converges to* $\dfrac{\alpha}{1-r}$.

Restated, the geometric series converges to $\frac{\alpha}{1-r}$ if $|r| < 1$, and diverges otherwise.

**Proof:** The proof requires some care, as the various cases contain their own technicalities.

- Case $r = 1$. In such a case, it is not difficult to see (we just count the terms!) that

$$S_N = \sum_{n=0}^{N} \alpha = (N+1)\alpha \longrightarrow \infty \qquad \text{as } N \to \infty.$$

  Thus $r = 1$ gives a divergent series.

- Case $r = -1$. In such a case, we have

$$\sum_{n=0}^{\infty} \alpha(-1)^n = \alpha - \alpha + \alpha - \alpha + \cdots ,$$

  and so
$$S_N = \begin{cases} \alpha, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

  In other words, $\{S_N\}_{N=0}^{\infty} = \alpha, 0, \alpha, 0, \alpha, 0, \cdots$, which is clearly a divergent sequence, i.e., the series itself is divergent (by definition).[5]

- Case $|r| > 1$. Here we can use the formula for the partial sums:

$$S_N = \frac{\alpha\left(1 - r^{N+1}\right)}{1-r}.$$

  Now there is only one term which is not a fixed constant, and so the convergence of this expression depends upon only the convergence that, $r^{N+1}$-term. Clearly if $r > 1$, this is an exponential growth, and diverges. For the general case $|r| > 1$, we get that[6]

$$r^{N+1} \text{ converges} \implies \left|r^{N+1}\right| \text{ converges} \iff |r|^{N+1} \text{ converges.} \qquad (10.13)$$

[5] Recall that the convergence of the series is defined by the convergence of the (sequence of) partial sums.
[6] This follows from continuity of the function $x \mapsto |x|$ giving us the "$\implies$." See Theorem 3.10.2, page 255.

But for $|r| > 1$, we have $|r|^{N+1}$ diverges, so with the contrapositive of (10.13) we have

$$\begin{aligned} |r| > 1 &\implies |r|^{N+1} \text{ diverges} \implies r^{N+1} \text{ diverges} \\ &\implies S_N = \frac{\alpha\left(1-r^{N+1}\right)}{1-r} \text{ diverges} \iff \sum_{n=0}^{\infty} \alpha r^n \text{ diverges.} \end{aligned}$$

- Case $|r| < 1$. Again we look at the variable part of the formula for $S_N$. It is enough to show that $|r| < 1 \implies r^{N+1}$ converges. One method is to use the sandwich theorem. In the argument below, note that $|r| < 1 \implies |r| \in (0,1) \implies |r|^{N+1} \longrightarrow 0$. The relevant sandwich theorem application is then (as $N \to \infty$):[7]

$$\underbrace{-|r|^{N+1}}_{\downarrow\, 0} = -\left|r^{N+1}\right| \quad \le \quad r^{N+1} \quad \le \quad \left|r^{N+1}\right| = \underbrace{|r|^{N+1}}_{\downarrow\, 0}$$

Thus $|r| < 1 \implies r^{N+1} \longrightarrow 0$ as $N \to \infty$. We can conclude that

$$|r| < 1 \implies S_N = \frac{\alpha\left(1-r^{N+1}\right)}{1-r} \longrightarrow \frac{\alpha(1-0)}{1-r} = \frac{\alpha}{1-r} \text{ (as } N \to \infty).$$

This completes the proof.

The implication above is worth repeating in a summarized form:

$$|r| < 1 \implies \sum_{n=0}^{\infty} \alpha r^n = \frac{\alpha}{1-r}. \tag{10.14}$$

**Example 10.1.9** *Here are some series computations using the theorem and (10.14).*

- $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 2 \cdot \frac{3}{2} = 3.$ $\qquad$ *($\alpha = 2$, $r = \frac{1}{3}$.)*
- $\displaystyle\sum_{n=0}^{\infty} 0.99^n = \frac{1}{1-0.99} = \frac{1}{.01} = 100.$ $\qquad$ *($\alpha = 1$, $r = 0.99$.)*
- $\displaystyle\sum_{n=0}^{\infty} 1.01^n$ *diverges.* $\qquad$ *($\alpha = 1$, $r = 1.01$ so $|r| > 1$, and the series diverges.)*
- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{9} \cdot \frac{3}{2} = \frac{1}{6}.$ $\qquad$ *(First term is $\alpha = \frac{1}{9}$, $r = \frac{1}{3}$.)*
- $\displaystyle 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \frac{1}{1-\left(\frac{-1}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$ $\qquad$ *($\alpha = 1$, $r = -\frac{1}{2}$.)*

[7] Recall that for any $x \in \mathbb{R}$, we have $-|x| \le x \le |x|$.

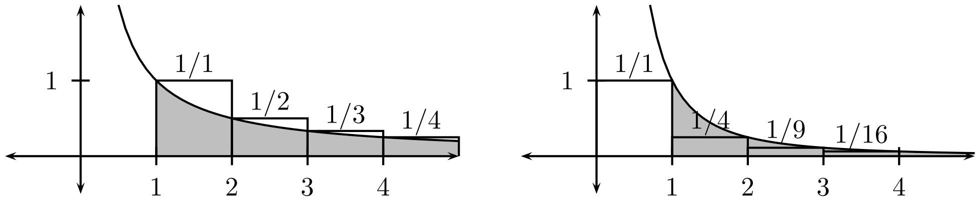
- $\displaystyle\sum_{n=1}^{\infty} e^{-n} = \frac{e}{1-\frac{1}{e}} = \frac{e}{1-\frac{1}{e}} \cdot \frac{e}{e} = \frac{e^2}{e-1}.$ $\qquad$ *($\alpha = e$, $r = \frac{1}{e}$.)*
- $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^{2n}} = \sum_{n=1}^{\infty} \frac{5}{9^n} = \frac{5/9}{1-\frac{1}{9}} = \frac{5/9}{8/9} = \frac{5}{8}.$ $\qquad$ *($\alpha = \frac{5}{9}$, $r = \frac{1}{9}$.)*

## Exercises

1. Show that the following series can be written as a telescoping series, and discuss its convergence:

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

2. Give an alternative proof of the formula (10.12) for the partial sums of geometric series. For this new proof, begin with the formula for $S_N$ as in the original proof (page 806), and then multiply by $(1-r)$, noting how the right-hand side simplifies. (See also page 87)




$$S_N \geq \int_1^{N+1} \frac{1}{x}\,dx = \ln(N+1) \longrightarrow \infty \qquad S_N \leq 1 + \int_1^{N} \frac{1}{x^2}\,dx = 2 - \frac{1}{N} \longrightarrow 2$$

**Figure 10.1:** Illustrations for showing that $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges. We represent the terms of the series to be summed as areas of rectangles, and show how they can be, respectively, underestimated or overestimated by areas under their respective curves. See the explanation following Example 10.2.1.

## 10.2 NTTFD and Integral Test

Because it is the exceptional case (e.g., geometric, telescoping) that we can actually find a compact formula for $S_N$, we have to develop other tests for the convergence or divergence of series. There will be several such tests, and which particular test or tests are expeditious and conclusive will vary from series to series. We explore the first of those tests in this section. We start with two series that are similar, though one converges and the other diverges.

**Example 10.2.1** *The following are facts regarding two particular series.*

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

These are not obvious at all, but need to be proven. We do so by comparing the terms in the series—represented by areas of rectangles—to certain improper integrals, as in Figure 10.1.

For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, we observe:

$$\begin{aligned}
S_1 &= \frac{1}{1} && \geq \int_1^2 \frac{1}{x}\,dx \\
S_2 &= \frac{1}{1} + \frac{1}{2} && \geq \int_1^3 \frac{1}{x}\,dx \\
S_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} && \geq \int_1^4 \frac{1}{x}\,dx \\
&\ \vdots && \ \vdots \\
S_N &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} && \geq \int_1^{N+1} \frac{1}{x}\,dx.
\end{aligned}$$

Thus $S_N \ge \int_1^{N+1} \frac{1}{x}\,dx = \ln(N+1) - \ln 1 = \ln(N+1) \to \infty \qquad \text{as } N \to \infty.$

We must conclude that $S_N \to \infty$ as $N \to \infty$, which implies that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (to infinity). In fact for this example we have strict inequality "$>$" in each of the above, but it is enough that we have the non-strict "$\ge$."

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, instead we note that

$$\begin{aligned}
S_2 &= \frac{1}{1} + \frac{1}{4} & &\le 1 + \int_1^2 \frac{1}{x^2}\,dx \\
S_3 &= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} & &\le 1 + \int_1^3 \frac{1}{x^2}\,dx \\
S_4 &= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} & &\le 1 + \int_1^4 \frac{1}{x^2}\,dx \\
&\vdots & &\vdots \\
S_N &= \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{N^2} & &\le 1 + \int_1^N \frac{1}{x^2}\,dx.
\end{aligned}$$

Now clearly $\{S_N\}_{N=1}^{\infty}$ is an increasing sequence, and $\int_1^N \frac{1}{x^2}\,dx$ is obviously increasing with $N$. Furthermore, for all $N$

$$S_N \le 1 + \int_1^N \frac{1}{x^2}\,dx \le 1 + \int_1^{\infty} \frac{1}{x^2}\,dx. \tag{10.15}$$

We can compute this improper integral as follows:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{\beta\to\infty} \int_1^{\beta} \frac{1}{x^2}\,dx = \lim_{\beta\to\infty} \left.\frac{-1}{x}\right|_1^{\beta} = \lim_{\beta\to\infty} \left(\frac{-1}{\beta} + 1\right) = 1. \tag{10.16}$$

Putting computation (10.16) into (10.15), we get

$$S_N \le 1 + 1 = 2.$$

Thus $\{S_N\}_{N=1}^{\infty}$ is a bounded and increasing sequence, and therefore converges, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N\to\infty} S_N$$

also converges, q.e.d. In fact, we even have an upper bound for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2.$$

However, we do not know from these computations its actual value. We only know that it does converge.[8]

[8] Note that we used an improper integral to investigate the convergence or divergence of the series, but we did not compute the series' exact value. We can get an *estimate*, meaning that we find a bound for it, but we can not compute the exact value with these methods. There are methods for doing so, but they are indirect and beyond the scope of this text. However, we can find how many terms must be added to be within a certain tolerance of the actual series' value, as we will see in a later subsection.

### 10.2.1 First Divergence Test: NTTFD

When we compare the two series above, we see that both have terms that are shrinking to zero, but one's series terms, $1/n^2$ shrink much faster than the other's, namely $1/n$, just as $\frac{1}{x^2}$ shrinks much faster than $\frac{1}{x}$, hence the former's improper integral on $[1,\infty)$ converges while the latter's does not. In fact, according to our next theorem it is necessary for the terms to shrink if the series is to have any chance of converging. However, *it is not enough that they shrink to zero; they have to shrink* **fast enough** *(assuming all terms are the same sign), or the series will diverge.*

Thus it is necessary (even though not sufficient) that the terms of a series must shrink to zero if the series is to have any chance of converging. This we codify in a theorem.

**Theorem 10.2.1** *Given a series* $\displaystyle\sum_{n=k}^{\infty} a_n$. *If the series converges, then* $a_n \longrightarrow 0$ *as* $n \to \infty$*:*

$$\left[\sum_{n=k}^{\infty} a_n \text{ converges}\right] \implies \left[\lim_{n\to\infty} a_n = 0\right]. \tag{10.17}$$

**<u>Proof</u>:** Suppose $\displaystyle\sum_{n=k}^{\infty} a_n$ converges, i.e., $\displaystyle\sum_{n=k}^{\infty} a_n = L$ for some $L \in \mathbb{R}$ (and in particular $L$ is finite). Then by definition

$$S_N \longrightarrow L \text{ as } N \to \infty.$$

Now recall $S_n = a_n + S_{n-1}$, so that $a_n = S_n - S_{n-1}$. Taking $n \to \infty$ we get

$$a_n = S_n - S_{n-1} \longrightarrow L - L = 0, \qquad \text{q.e.d.}$$

It should be intuitively clear that, if we are going to "add up" infinitely many terms, and have the sums approach a finite number, then the terms we are adding are going to have to shrink to zero. The proof uses the fact that $S_n \to L \implies S_{n-1} \to L$, the latter limit occurring "one step behind" the former, but occurring nonetheless since $n \to \infty \implies n-1 \to \infty \implies S_{n-1} \to L$.

Note that it was important that $L$ be finite in the limit computation above. For instance, if $S_n \to \infty$, we would have $a_n = S_n - S_{n-1}$ giving $\infty - \infty$-form (which is indeterminate) as $n \to \infty$.

Again, the intuition behind the theorem is that, in order to be able to add infinitely many terms—one at a time in the sense that we compute $S_k$, $S_{k+1}$, $S_{k+1}$, etc., and look for a trend in these sums towards $L$—the terms that we add, i.e., $a_k$, $a_{k+1}$, $a_{k+2}$, etc., have to shrink eventually if the partial sums are to approach a finite number $L$.

In fact, the form of the theorem which we use is the contrapositive. Recall the logical equivalence $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$.[9] In this case, $P$ is the statement that the series converges (to a finite number $L$), while $Q$ is the statement that $a_n \to 0$. The contrapositive for of Theorem 10.2.1 is our main result in this section, and we dub that result *nth term test for divergence*, or NTTFD:

[9] To be sure, here $P \longrightarrow Q$ is read, "$P$ implies $Q$." The symbol "$\sim$" is still the "not," or logical negation, operator. The symbol "$\iff$" stands in for logical equivalence. Recall $P \longrightarrow Q$ and $(\sim Q) \longrightarrow (\sim P)$ are *contrapositives* of each other, and are logically equivalent.

**Theorem 10.2.2 (NTTFD)** *If it is* **not** *the case that* $a_n \to 0$*, then* $\sum_{n=k}^{\infty} a_n$ *diverges. Put symbolically,*

$$a_n \not\to 0 \implies \sum_{n=k}^{\infty} a_n \text{ diverges.} \tag{10.18}$$

**<u>Proof</u>:** It is enough to say that this is the contrapositive of Theorem 10.2.1, and therefore also true. One way to write this symbolically is the following.

$$\underbrace{\sum_{n=k}^{\infty} a_n \text{ converges } \longrightarrow (a_n \to 0)}_{\text{always true by Theorem 10.2.1}} \iff \underbrace{[\sim (a_n \to 0)] \longrightarrow \sum_{n=k}^{\infty} a_n \text{ diverges}}_{\text{statement of Theorem 10.2.2}}.$$

The statement on the right must be a tautology (always true), since it is equivalent to the statement on the left, which—being the statement of Theorem 10.2.1—is itself a tautology. The statement on the right being a tautology means that it can stand alone as a tautology, as (10.18), so "[ ] $\implies \sum$" can replace "[ ] $\longrightarrow \sum$," q.e.d.

This theorem is undoubtedly one of the most misunderstood and misapplied results in all of Calculus I and II. It is as important to understand what it does not say, as it is to understand what it says. The theorem says that if the terms of a series do not shrink to zero, then the series must diverge.

But it is not as comprehensive as one might think. After all, if the terms of a series *do* shrink to zero, the theorem is silent! (Therein lies the unfortunately very common mistake made by calculus students.) To emphasize this we look at the following examples.

**Example 10.2.2** *Discuss what Theorem 10.2.2 has to say about the series*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \cos\frac{1}{n}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$;

(e) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$.

<u>*Solution:*</u>

*(a)* $\displaystyle\frac{n}{2n+1} \longrightarrow \frac{1}{2} \neq 0 \overset{\text{NTTFD}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{n}{2n+1}$ *diverges.*

*(b)* $\displaystyle\frac{1}{n+1} \longrightarrow 0.$ *The NTTFD is inconclusive.*

*(c)* $\displaystyle\cos\frac{1}{n} \longrightarrow \cos 0 = 1 \neq 0 \overset{\text{NTTFD}}{\Longrightarrow} \sum_{n=1}^{\infty} \cos\frac{1}{n}$ *diverges.*

*(d)* $\sin\dfrac{1}{n} \longrightarrow \sin 0 = 0$. *The NTTFD is inconclusive.*

*(e)* $\sin\dfrac{1}{n^2} \longrightarrow \sin 0 = 0$. *The NTTFD is inconclusive.*

Looking closely at the symbolic statement of NTTFD given in (10.18), we see that there is never an implication of convergence. Indeed, the test either concludes divergence, or is inconclusive. This is a very quick but incomplete test, which can only detect divergence in certain (still common) circumstances, namely that $a_n \not\to 0$.

Indeed, in (a) and (c) above, NTTFD gave us divergence. However, it said nothing in (b), (d) and (e), as the "if" part of the theorem was not true. In fact, of these three in which NTTFD is silent—(b), (d) and (e)—it turns out that (*b*) and (*d*) are divergent, while (*e*) is convergent. The methods to see this are introduced in later sections. An example from the exercises in the last section gives a case where we can in fact show that it is possible that $a_n \to 0$, but the series diverges:

**Example 10.2.3** *Consider the series* $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$. *Determine if it converges or diverges.*

*Solution: First we note* $\ln\left(\frac{n}{n+1}\right) \longrightarrow \ln 1 = 0$ *as* $n \to \infty$. *Thus NTTFD is inconclusive.*[10]

*Looking closer at this series, we should eventually notice that it is telescoping. This becomes clear if we rewrite it:*

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} [\ln(n) - \ln(n+1)] = [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + [\ln 3 - \ln 4] + \cdots .$$

*It is not difficult to see that* $S_N = \ln 1 - \ln N = -\ln N$, *and thus*

$$S_N = -\ln N \longrightarrow -\infty \text{ as } N \to \infty.$$

*Since the partial sums diverge, by definition so does the series.*

### 10.2.2 Integral Test Proper (IT)

Of course there are series which converge. However the NTTFD is never powerful enough to ever prove it (but can sometimes detect divergence in a series). Proving actual convergence requires other tests. The integral test is one such test:

**Theorem 10.2.3** *Suppose we have a series* $\displaystyle\sum_{n=k}^{\infty} a_n$ *such that*

1. $a_n = f(n)$ *for each* $n \geq k$, *where*
2. $f(x) \geq 0$ *is continuous and nonincreasing on* $[k, \infty)$.

*Then* $\displaystyle\sum_{n=k}^{\infty} a_n$ *and* $\displaystyle\int_k^{\infty} f(x)\,dx$ *both converge or both diverge. In other words,*

$$\sum_{n=k}^{\infty} a_n \text{ converges} \iff \int_k^{\infty} f(x)\,dx \text{ converges.}$$

[10]Some textbooks would write that the test "fails." That seems a bit strong. It is merely inconclusive, so we need to look deeper at the particular series and perhaps employ some other test which will be conclusive.

*Equivalently,*

$$\sum_{n=k}^{\infty} a_n \text{ diverges} \iff \int_k^{\infty} f(x)\,dx \text{ diverges.}$$

**Proof:** We will not write the whole proof here, but just mention that it follows the same kind of reasoning we used to show $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges. It is not too difficult to see—by drawings similar to those early examples—that

(a) If $\int_k^{\infty} f(x)\,dx$ diverges, then it diverges to $\infty$, and so as $N \to \infty$ we have

$$S_N \geq \int_k^{N+1} f(x)\,dx \to \infty,$$

so $\sum_{n=k}^{\infty} a_n$ diverges to infinity.

(b) If $\int_k^{\infty} f(x)\,dx$ converges and is thus finite, then

$$S_N \leq a_k + \int_k^N f(x)\,dx \leq a_k + \int_k^{\infty} f(x)\,dx < \infty,$$

so $S_N$ is a bounded, clearly nondecreasing sequence, so it must converge, implying (by definition) that $\sum_{n=k}^{\infty} a_n$ converges.

**Example 10.2.4** *Consider any series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$*, where* $p > 0$*. Then* $f(x) = 1/x^p$ *is clearly a decreasing function on* $[1, \infty)$.[11] *Now*

$$\lim_{\beta\to\infty} \int_1^{\beta} x^{-p}\,dx = \begin{cases} \displaystyle\lim_{\beta\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^{\beta} = \frac{-1}{1-p} & \text{if } p > 1 \quad (1-p<0) \\ \displaystyle\lim_{\beta\to\infty} \left.\ln x\right|_1^{\beta} = \infty & \text{if } p = 1 \\ \displaystyle= \lim_{\beta\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^{\beta} = \infty & \text{if } p < 1 \quad (1-p>0) \end{cases}$$

*from which we get that* $\int_1^{\infty} \frac{1}{x^p}\,dx$ *converges if and only if* $p > 1$*, and diverges if and only if* $p \leq 1$*. This gives us the following as well:*

**Theorem 10.2.4** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$ *(and diverges if and only if* $p \leq 1$*).*

Series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ are called *p-series.* They are nearly as important as geometric series within the general theory of series. If one forgets about the two cases, the integral test, Theorem 10.2.3, page 817 makes deriving this last theorem fairly straightforward.

[11]We can note that the denominator is increasing and positive, and so intuitively these fractions $1/x^p$ must shrink. On the other hand, we can also compute $f'(x) = \frac{d}{dx}x^{-p} = -p \cdot x^{-p-1} = -p/(x^{p+1}) < 0$, so the fact that $f' < 0$ on $[1, \infty)$ also shows that $f(x)$ is decreasing there.

**Example 10.2.5** *Determine if the series* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ *converges.*

*Solution: Clearly this is a series of positive terms, which decrease monotonically. We can look therefore at* $\int_2^\infty \frac{1}{x\ln x}\,dx$*, since* $f(x)=1/(x\ln x)$ *is also positive and decreasing on* $[2,\infty)$*. Now*

$$\lim_{\beta\to\infty}\int_2^\beta \frac{1}{x\ln x}\,dx=\lim_{\beta\to\infty}\ln|\ln x|\big|_2^\beta=\lim_{\beta\to\infty}\left[\ln(\ln\beta)-\ln(\ln 2)\right]=\infty,$$

*we conclude that* $\int_2^\infty\frac{1}{x\ln x}\,dx$ *diverges to infinity, and therefore so does the original series, by the Integral Test.*

We will have other tests in subsequent sections. However, for series which satisfy the hypotheses of the Integral Test, it is the most sensitive, and can determine convergence or divergence for many series for which other tests are inconclusive. (It already has shown convergence or divergence for series for which the NTTFD is silent.) It is also interesting to note that the Integral Test, or its general reasoning buried in the proof, can determine which geometric series converge and which diverge, assuming $\alpha>0$, $r>0$, and $r\neq 1$ (or the series obviously diverges).

$$\lim_{\beta\to\infty}\int_0^\beta \alpha r^x\,dx=\lim_{\beta\to\infty}\left.\frac{\alpha r^x}{\ln r}\right|_0^\beta\longrightarrow\begin{cases}\infty & \text{if} \quad r>1\\ \frac{-\alpha}{\ln r} & \text{if} \quad r\in(0,1).\end{cases}$$

Note that for $r\in(0,1)$ we have $\ln r<0$ so the integral is positive. So the Integral Test concludes that $\sum\alpha r^n$ converges if $r\in(0,1)$ and diverges if $r\geq 1$. (If $\alpha<0$ we just look at $\sum\alpha r^n=-\sum(-\alpha)r^n$, and if $\alpha=0$ the series obviously converges.)

**Example 10.2.6** *Consider* $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}$*, and discuss if it converges or diverges.*

*Solution: First note that* $a_n=f(n)$ *where* $f(x)=\frac{1}{x^2+1}$ *is a decreasing function on* $[0,\infty)$*. Again this is intuitive, but a quick check shows* $f'<0$ *on* $(0,\infty)$*, which is enough to show* $f$ *is decreasing on* $[0,\infty)$*.*[12]

*Next we check the relevant improper integral.*

$$\int_0^\infty\frac{1}{x^2+1}\,dx=\lim_{\beta\to\infty}\int_0^\beta\frac{1}{x^2+1}\,dx=\lim_{\beta\to\infty}\left(\tan^{-1}\beta-\tan^{-1}0\right)=\frac{\pi}{2}-0=\frac{\pi}{2}.$$

*Since the improper integral converges, so does the series* $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ *converge.*

One interesting question is how many terms of the series do we need to add in order to get a good approximation of the full series? We clearly can not add infinitely many, and this series is not telescoping and not geometric, and finding a simple formula for $S_N$ for which we can let $N\to\infty$ seems unlikely. So instead we look back to the integral that proves the series converges. While there seems to be a general formula, it is probably best practice (unless one does such problems repeatedly) to re-draw the situation when needed. Nonetheless, it is interesting to see the general rule:

[12] In fact, we really only need $f(x)$ to be decreasing on some set $[k,\infty)$, since we can always sum the finitely many terms that occur before $n=k$. It is the infinite "tail-end" sum of the series that determines convergence, i.e., we need to compute if $\sum_{n=k}^\infty a_n$ converges. We have the liberty to ignore what happens for any finite number of terms at the "front" of the series, assuming they are defined.

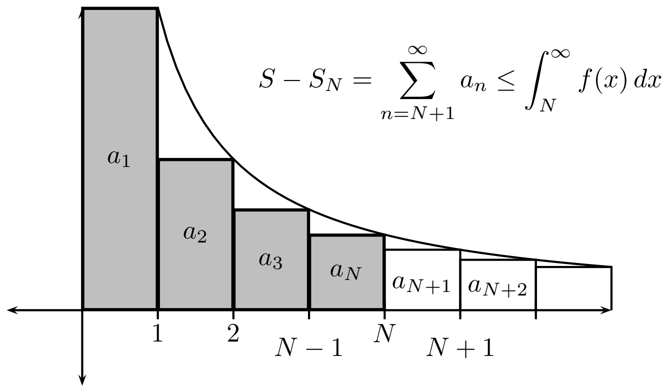



**Figure 10.2:** Here we look at the error $S - S_N$ if we sum only the first terms up to and including $a_N$. $S_N$ is represented by the areas of the colored rectangles; the remaining "tail-end" of the series, i.e., the error $S - S_N$ from missing the remaining terms, is represented by the remaining, unshaded rectangles, which are safely under the curve $y = f(x)$. (For simplicity of illustration we show the case $N = 4$, and the series starts at $n = 1$.)

**Theorem 10.2.5** *Suppose $\sum a_n$ satisfies the hypotheses of the Integral Test, in the sense that $a_n = f(n)$ for some $f(x)$ defined, nonnegative and nonincreasing on some interval $[k, \infty)$, and $\int_k^\infty f(x)\,dx < \infty$. Then the series converges to some value $\sum a_n = S$, and for $N > k$ we have*

$$|S - S_N| = S - S_N \le \int_N^\infty f(x)\,dx. \tag{10.19}$$

For our previous example, if we wished to know $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ to within 0.10, we can find $N$ so that $\int_N^\infty \frac{1}{x^2+1}\,dx \le 0.10$, and that will guarantee $S - S_N \le 0.10$ as well. Now for any given $N$, we have

$$\int_N^\infty \frac{1}{x^2+1}\,dx = \lim_{\beta\to\infty} \int_N^\beta \frac{1}{x^2+1}\,dx = \lim_{\beta\to\infty} \left(\tan^{-1}\beta - \tan^{-1} N\right) = \frac{\pi}{2} - \tan^{-1} N.$$

For $\pi/2 - \tan^{-1} N \le 0.01$, we need

$$\tan^{-1} N \ge \frac{\pi}{2} - 0.10 \implies N \ge \tan\left(\frac{\pi}{2} - 0.10\right) \approx 9.966644419,$$

so we take $N \ge 10$ for such accuracy.

If instead we wanted accuracy to within 0.001, we have a similar calculation yielding

$$\tan^{-1} N \ge \frac{\pi}{2} - 0.001 \implies N \ge \tan\left(\frac{\pi}{2} - 0.001\right) \approx 999.99962177,$$

so we would need to take $N \ge 1000$, and thus compute

$$\sum_{n=0}^{1000} \frac{1}{n^2+1}$$

to assure that we achieve such accuracy in using $S_N$ to approximate the full series $S$. Note that it quickly becomes more suitable for numerical (specifically, electronic) computational devices.[13]

[13] In computing the inequality, we used that $\tan x$ is an increasing function on $(-\pi/2, \pi/2)$, so for all $x_1, x_2 \in (-\pi/2, \pi/2)$, we have $x_1 < x_2 \implies \tan x_1 < \tan x_2$.

In a later section, we will note that the convergence of $\sum \frac{1}{n^2}$ guarantees the convergence of $\sum \frac{1}{n^2+1}$, because

$$0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2},$$

and since $1/n^2$ is *summable*, i.e., $\sum \frac{1}{n^2}$ converges ($p$-series, $p > 1$), the series of smaller terms must also. That is called the *Direct Comparison Test*, though there are still others.[14]

# Exercises

1. Use the NTTFD to determine which of the following series must diverge (based only upon that test). If NTTFD is inconclusive, so state.

   (a) $\displaystyle\sum_{n=1}^{\infty} e^{1/n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

2. Use the integral test to determine convergence or divergence of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

3. It is known from other fields that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6.$

   (a) Find $S_{10}$, and estimate how accurate that is using the integral test estimate (10.19).

   (b) How large would $N$ need to be to ensure that $S_N$ is within 0.0001 of the full series?

4. The integral test requires that $f(x)$ be eventually monotonically decreasing. Here we give an example where the series and improper integral behave very differently, to show that we can not relax our hypotheses on $f$.

   (a) Show that $\displaystyle\int_1^{\infty} \sin^2(\pi x)\, dx$ diverges (to infinity).

   (b) Show that $\displaystyle\sum_{n=1}^{\infty} \sin^2(\pi \cdot n)$ converges.

   (c) Graph $y = \sin^2(\pi x)$, for $x \geq 0$. Explain why this is not a proper function to choose when determining the convergence of $\sum \sin(n\pi)$ with the integral test.

[14] As is often the case, we use "$\leq$" when we in fact have "$<$." This is partly to show that "$\leq$" is sufficient here, though to be more precise we should use "$<$."

## 10.3 Comparison Tests

Before proceeding to the topic of this section, we briefly review what we have developed so far regarding series.

To say that $\sum a_n$ converges to some number $S \in \mathbb{R}$ is, by definition, to say that $S_N \longrightarrow S$, where $S_N$ is the $N$th partial sum of the series. There were two series for which we were able to compute $S_N$ and therefore its limit as $N \to \infty$: the telescoping series and the geometric series.

Then we realized that there are many series which we can prove converge, as in cases where $\{S_N\}$ is a bounded and increasing sequence, but we could not directly calculate the infinite sum. We proved these converged using integral tests, where we compared them to $\int_1^\infty f(x)\,dx$, where $a_n = f(n)$. The series converged if and only if the improper integral did, but we were careful that, (1) $f(x)$ was eventually nonincreasing on $[1, \infty)$ or the relevant interval, and (2) we did not make the mistake of claiming that the series and the integral converged to the same number, but only that their behaviors were similar enough that convergence (or divergence) of one meant the same for the other.

After these developments, we made special note of two types of series for which we know immediately whether or not they converge:

1. geometric series: $\displaystyle\sum_{n=0}^{\infty} \alpha r^n$ converges iff $r \in (-1, 1)$, diverges otherwise (assuming $\alpha \neq 0$);

2. $p$-series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$, diverges otherwise (i.e., for $p \leq 1$).

For the $p$-series, we proved this using the Integral Test, and gave some special mention to the borderline case, $p = 1$, mentioning that the *harmonic series* $\sum \frac{1}{n}$ diverges.

These two series types above will be very useful for both of our comparison tests developed here, especially the $p$-series. Before we look at these, however, we make another observation:

**Theorem 10.3.1** $\displaystyle\sum_{n=1}^{\infty} a_n \textit{converges} \iff (\forall M \in \{1, 2, 3, \cdots\}) \left[\sum_{n=M}^{\infty} a_n \ \textit{converges}\right]$

This is just the statement that the first $M - 1$ terms are not what determine convergence or divergence, no matter how large or small $M$ happens to be. It is the *tail end* of the series, which contains infinitely many terms to sum, which determines if the series converges or not. We can always add a finite number of real numbers and the result will be a finite real number, but when we attempt to somehow "add" an infinite number of terms, in truth we can not but instead appeal to the well-defined partial sums $S_N$, and then use a limit argument to let $N \to \infty$.

This theorem is useful because many tests for convergence or divergence in fact allow us to ignore a finite number of terms in the series, to focus instead on the crucial "tail end" of the series.

Recall that if a series $\sum a_n$ is convergent, we say that the sequence of terms $a_n$ is *summable.* (It is also common to say the convergent series itself is "summable.") It is helpful to have such vocabulary at our disposal to streamline later arguments.

### 10.3.1 Direct Comparison Test (DCT)

This test, like the Integral Test, is a test for series $\sum a_n$ and $\sum b_n$ of positive-term series. Here is its statement:

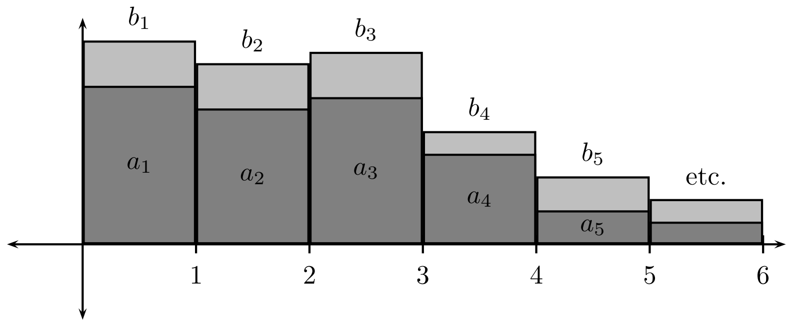



**Figure 10.3:** Illustration for the Direct Comparison Test (DCT). The rectangles whose areas represent the terms $a_n$ are inside the taller rectangles whose areas represent the terms $b_n$. For simplicity we illustrate the case $0 < a_n < b_n$, though the test works for the nonstrict case $0 \le a_n \le b_n$ as well. Clearly in each case we have $0 \le \sum_{n=1}^N a_n \le \sum_{n=1}^N b_n$, so intuitively if $\sum b_n$ converges so must $\sum a_n$. The contrapositive is then also true: if $\sum a_n$ diverges then so must $\sum b_n$. (Recall $P \longrightarrow Q \iff (\sim Q) \longrightarrow (\sim P)$.)

**Theorem 10.3.2 Direct Comparison Test (DCT)** *Suppose that $0 \le a_n \le b_n$ for all $n = 1, 2, 3, \cdots$. If (the larger series) $\sum b_n$ converges, so does $\sum a_n$. However, if (the smaller series) $\sum a_n$ diverges, so does $\sum b_n$.*

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{10.20}$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \implies \sum_{n=1}^{\infty} b_n \text{ diverges.} \tag{10.21}$$

Perhaps the easiest way to interpret this is as follows:

> If the series with the larger terms converges, so does the series with the smaller terms; if the series with the smaller terms diverges, so must the series with the larger terms.

To be precise, this is true assuming that all terms are nonnegative. To be even more precise, we should then also use "greater" for "larger," and "lesser" for "smaller" above. See Figure 10.3.

**Proof:** First we show that $\sum b_n$ converges implies $\sum a_n$ converges. To show this, we note that if $S_N$ is the $N$th partial sum of $\sum a_n$, then clearly $S_N$ is nondecreasing (since $a_n \ge 0$), and furthermore

$$S_N = a_1 + a_2 + \cdots + a_N \le b_1 + b_2 + \cdots + b_n \le \sum_{n=1}^{\infty} b_n,$$

so $S_N \le \sum b_n$, i.e., each $S_N$ is bounded by the whole series $\sum b_n$. Thus $\{S_N\}_{N=1}^{\infty}$ is a bounded, nondecreasing sequence, and must therefore converge. This proves (10.20). In fact (10.21) is just its contrapositive so it is also then proven, but it is also interesting to prove it separately.

So suppose instead that $\sum a_n$ diverges. Being a nonnegative-term series it must therefore diverge to infinity. So let $S_N$ be the $N$th partial sum of $\sum a_n$ and $\mathscr{S}_N$ be the $N$th partial sum of $\sum b_n$. Then

$$\mathscr{S}_N = b_1 + b_2 + \cdots + b_N \geq a_1 + a_2 + \cdots + a_N = S_N \longrightarrow \infty \quad \text{as} \quad N \to \infty,$$

showing that $\mathscr{S}_N$ diverges to infinity ($\mathscr{S}_N \geq S_N \to \infty$), i.e., $\sum b_n$ diverges (to infinity), q.e.d.

Of course the same is true if the series start somewhere besides $n = 1$. Furthermore, in light of Theorem 10.3.1 (page 821), in fact we only require $0 \leq a_n \leq b_n$ to be true "eventually," i.e., for all $n \geq N$ where $N$ is some finite number. (Here we use $N$ differently than in the partial sums $S_N$.)

**Corollary 10.3.1** *Suppose for some $N$ we have $n \geq N \implies 0 \leq a_n \leq b_n$. Then* (10.20) *and* (10.21) *still hold.*

**Example 10.3.1** *Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln n + n^{3/2}}$. We note that for $n \geq 1$ we have $\ln n \geq 0$, so the denominator is greater than or equal to $n^{3/2}$:*

$$0 < n^{3/2} \leq \ln n + n^{3/2} \implies 0 < \frac{1}{\ln n + n^{3/2}} \leq \underbrace{\frac{1}{n^{3/2}}}_{\textit{summable}}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges} \implies \sum_{n=1}^{\infty} \frac{1}{\ln n + n^{3/2}} \text{ converges.}$$

*(The first inequalities "$0 <$" are just to indicate that all terms are positive, so the Direct Comparison Test, DCT, applies.) Since $\sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2 > 1$), we can conclude by the DCT that $\sum \frac{1}{\ln n + n^{3/2}}$ also converges.*

Note that we used the fact that, when all terms are positive, a larger denominator implies a smaller fraction.[15] We also used that $\ln n \geq 0$ for $n \geq 1$ (though "eventually", i.e., for all $n \geq N$ for some $n$, is enough).

**Example 10.3.2** *Consider $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. While we could use an integral test on this series to determine convergence or divergence,[16] it will be faster to note that, for large enough $n$ (in particular,*

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[15] Recall that if $A, B, C, D > 0$, then

$$A < B \implies \frac{A}{C} < \frac{B}{C}, \qquad \text{(larger numerator} \implies \text{larger fraction)}$$
$$C < D \implies \frac{A}{C} > \frac{A}{D}, \qquad \text{(larger denominator} \implies \text{smaller fraction)}$$

[16] We might also consider the NTTFD, Theorem 10.2.2 (page 815), but we would note that

$$\lim_{n\to\infty} \frac{\ln n}{n} \overset{\infty/\infty}{\underset{\text{LHR}}{=\!=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \overset{(1/\infty)/1}{=\!=} 0,$$

so the NTTFD is inconclusive. However it is also not difficult to observe that

$$\int \frac{\ln x}{x}\,dx = \lim_{\beta\to\infty} \int_1^{\beta} \frac{\ln x}{x}\,dx = \lim_{\beta\to\infty} \frac{1}{2}(\ln x)^2 \Big|_1^{\beta} = \lim_{\beta\to\infty} \left[\frac{1}{2}(\ln \beta)^2 - 0\right] = \infty,$$

which proves $\sum \frac{\ln n}{n}$ diverges.

*$n > e^1$ so for integers, $n \geq 3$) we have*

$$0 < \underbrace{\frac{1}{n}}_{\text{not summable}} < \frac{\ln n}{n},$$

$$\therefore \quad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{smaller terms}} \text{ diverges} \implies \underbrace{\sum_{n=1}^{\infty} \frac{\ln n}{n}}_{\text{larger terms}} \text{ diverges.}$$

The DCT is particularly useful when we can not (or can not easily) integrate the respective function, as in our first example (Example 10.3.1, page 823).

**Example 10.3.3** *Discuss the convergence/divergence of* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$.

*Solution: We note that the terms in the series would be larger if the "+1" were absent in the denominators.*

$$0 \leq \frac{1}{\sqrt{n^5+1}} < \frac{1}{\sqrt{n^5}} = \underbrace{\frac{1}{n^{5/2}}}_{\text{summable}} \tag{10.22}$$

$$\therefore \quad \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}}_{\text{larger terms}} \text{ converges} \implies \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}}_{smallerterms} \text{ converges.} \tag{10.23}$$

### 10.3.2 A Hierarchy of Functions and DCT

It is useful to note that different functions $f(n)$ which grow to $\infty$ as $n \to \infty$ do so much faster than others.

**Theorem 10.3.3** *In the list below, any function $f(n)$ listed to the left of another function $g(n)$ will grow so much more slowly than $g(n)$ that $\lim_{n\to\infty} f(n)/g(n) = 0$.*

$$\ln(\ln n), \quad \ln n, \quad n^r (r > 0), \quad n^s (s > r), \quad a^n (a > 1), \quad b^n (b > 1), \quad n!, \quad n^n.$$

This is not an exhaustive list, but offers some useful facts and intuition. It will take some effort and later methods to show why $n! = 1 \cdot 2 \cdot 3 \cdots n$ fits into the hierarchy where it does. It is somewhat more intuitive to see that $n^n$ is properly placed, at least in comparison to $n!$:

$$\begin{array}{ll}
0! = 1 \text{ (by definition)} & \\
1! = 1 & 1^1 = 1 \\
2! = 1 \cdot 2 = 2 & 2^2 = 2 \cdot 2 = 4 \\
3! = 1 \cdot 2 \cdot 3 = 6 & 3^3 = 3 \cdot 3 \cdot 3 = 27 \\
4! = 1 \cdot 2 \cdot 3 \cdot 4 = 4 \cdot 3! = 24 & 4^4 = 4 \cdot 4 \cdot 4 \cdot 4 = 256 \\
5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120 & 5^5 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 3,125
\end{array}$$

and so on. This hierarchy helps us to use the DCT to determine convergence or divergence of some series, and sometimes the NTTFD to determine divergence.

**Example 10.3.4** *Consider* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$. *We can direct-compare this to* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$. *(Note why we can not start at $n=1$.) For larger enough $n$ (actually for all $n>0$), we have $\ln n < n$ (proven by the fact that $(\ln n)/n \to 0$, see original limit in Footnote 16, page 823), and so for large enough $n$ we have*

$$0 < \underbrace{\frac{1}{n}}_{\text{not summable}} < \frac{1}{\ln n}$$

$$\therefore \sum_{n=2}^{\infty}\frac{1}{n} \text{ diverges} \implies \sum_{n=2}^{\infty}\frac{1}{\ln n} \text{ diverges.}$$

**Example 10.3.5** *Consider* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$. *Since for instance $n! > n^2$ for large enough $n$ (in particular, $n \geq 4$), we can write*

$$0 < \frac{1}{n!} < \underbrace{\frac{1}{n^2}}_{\text{summable}}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} \text{ converges} \implies \sum_{n=1}^{\infty}\frac{1}{n!} \text{ converges.}$$

Note that not all comparisons are useful. For instance, it is true that $1/n! < 1/n$, but $1/n$ is not summable, i.e., $\sum \frac{1}{n}$ diverges, so that comparison does not let us conclude convergence or divergence of the series $\sum \frac{1}{n!}$. There are series that converge and series that diverge whose terms are smaller than those in the harmonic series $\sum \frac{1}{n}$. However there are no series with terms greater than the harmonic series' terms but that converge.

Note also we could have used the hierarchy of functions to conclude $1/n! < 1/2^n$, and $\sum(1/2)^n$ converges (geometric, $|r| = 1/2 < 1$), also giving us $\sum \frac{1}{n!}$ converges by the DCT.

**Example 10.3.6** *Consider* $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^2 n}{\sqrt{n^3+5}}$. *Since $0 \leq \cos^2 n \leq 1$, we can write*

$$0 \leq \frac{\cos^2 n}{\sqrt{n^3+5}} \leq \frac{1}{\sqrt{n^3+5}} < \frac{1}{\sqrt{n^3}} = \underbrace{\frac{1}{n^{3/2}}}_{\text{summable}},$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \text{ converges} \implies \sum_{n=1}^{\infty}\frac{\cos^2 n}{\sqrt{n^3+5}} \text{ converges.}$$

**Example 10.3.7** *Consider* $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+1}$. *Note that*

$$0 < \frac{n}{n^4+1} < \frac{n}{n^4} = \underbrace{\frac{1}{n^3}}_{\text{summable}}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^3} \text{ converges} \implies \sum_{n=1}^{\infty}\frac{n}{n^4+1} \text{ converges.}$$

### 10.3.3 Limit Comparison Test (LCT)

At times, the DCT requires a bit more cleverness than necessary. For instance, if we take the previous example and make minor changes, considering instead $\displaystyle\sum_{n=2}^{\infty}\frac{n}{n^4-1}$, our intuition is that for large enough $n$, the terms in this series should be very similar to those of $\sum\frac{n}{n^4+1}$, since the difference in the denominators becomes less and less significant for large $n$. So our intuition is that this series probably converges as well, but finding a formula for $S_N$, or a series which has larger terms but still converges, or even using an integral test, can be difficult (or in the case of finding a formula for $S_N$, perhaps impossible). One comparison that does work—included here just for completeness—is the following:

$$0<\frac{n}{n^4-1}=\frac{n^2}{n(n^4-1)}<\frac{2(n^2-1)}{n(n^4-1)}=\frac{2}{n(n^2+1)}<\underbrace{\frac{2}{n^3}}_{\text{summable}}$$

However the first equality does not immediately seem well motivated, and the second inequality is not so obvious unless perhaps we write out several terms of each to see the pattern emerging.

And yet it seems like we should be able to argue that $\frac{n}{n^4-1}\approx\frac{n}{n^4}=\frac{1}{n^3}$ implies that, since $\sum\frac{1}{n^3}$ converges we can conclude that $\sum\frac{n}{n^4-1}$ also converges. Indeed we can, according to the following theorem.

**Theorem 10.3.4 Limit Comparison Test (LCT).** *Suppose $a_n\ge 0$, $b_n>0$ for large enough $n$, and that $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=L\in[0,\infty]$.*

*1. If $L\in(0,\infty)$, then*

$$\begin{aligned}\sum a_n \text{ converges} &\iff \sum b_n \text{ converges, or equivalently}\\ \sum a_n \text{ diverges} &\iff \sum b_n \text{ diverges.}\end{aligned}$$

*2. If $L=\infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

*3. If $L=0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*

The idea of the theorem can be more casually described as follows:

1. for large $n$, we have $a_n\approx L\cdot b_n$, and so if $L\in(0,\infty)$, both series converge or both diverge, since the "tail end" of one series is approximately a constant multiple ($L\ne 0,\infty$) of the "tail end" of the other;
2. if $L=\infty$, then $a_n$ is a larger and larger multiple of $b_n$, so if $\sum b_n$ diverges, so does $\sum a_n$;[17]
3. if $L=0$, then $a_n$ is a smaller and smaller multiple of $b_n$, so if $\sum b_n$ converges, so does $\sum a_n$.

A proof would actually rely on the DCT, by showing that we can eventually make the correct comparison with an appropriate series to give us the conclusion. Without going through every case, we look at the proof that if $\sum b_n$ converges, and $a_n/b_n\longrightarrow L\in(0,\infty)$, then we must conclude $\sum a_n$ converges.

[17] Technically, we have not defined what it means for $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$, so we mean that the sequence $\{a_n/b_n\}$ diverges to infinity. For the curious reader, a definition of $c_n\to\infty$ would be $(\forall M>0)(\exists N)[n>N\longrightarrow c_n>M]$.

**Proof:** Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty)$, and that $\sum b_n$ converges. Then for large enough $n$, we have $a_n/b_n \approx L$, so $a_n/b_n \in (L - \frac{1}{2}L, L + \frac{1}{2}L) = \left(\frac{1}{2}L, \frac{3}{2}L\right)$. Thus for large enough $n$ we have $0 < a_n < \frac{3}{2} \cdot b_n$, so we can direct-compare $\sum a_n$ to $\sum \frac{3}{2} \cdot b_n$ (which converges, since $\sum b_n$ does):

$$\sum b_n \text{ converges } \implies \sum \left(\frac{3}{2} \cdot b_n\right) = \frac{3}{2} \cdot \sum b_n \text{ converges } \implies \sum a_n \text{ converges,}$$

by the DCT. The other cases are proven similarly, with modifications.

The case proved above is the most commonly referenced case, and so we make the following definition:

**Definition 10.3.1** *For two positive-term series $\sum a_n$ and $\sum b_n$, we call the series limit-comparable if and only if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty).$$

Note we omit the cases $L \in \{0,\infty\}$. With this definition, part of the Limit Comparison Test (LCT) can now be phrased:

*Two limit-comparable series will both converge, or both diverge.*

**Example 10.3.8** *Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. We could use an integral test, but that would require either a complicated formula or trigonometric substitution to find the relevant antiderivative. The naïve Direct Comparison Test would have us note that $1/\sqrt{n^2+1} < 1/n$, but $\sum(1/n)$ is not summable, so being a "smaller" series than a divergent series is inconclusive.*

*However, for large enough $n$, it seems $1/\sqrt{n^2+1} \approx 1/\sqrt{n^2} = 1/n$, so the series seems similar to the harmonic series $\sum(1/n)$. So we verify that the two are limit-comparable:*

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} \stackrel{0/0}{\overline{\overline{\text{ALG}}}} \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \stackrel{\infty/\infty}{\overline{\overline{\text{ALG}}}} \lim_{n\to\infty} \frac{\cancel{n} \cdot 1}{\cancel{n} \cdot \sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1=0}} = 1 \in (0,\infty).$$

*Since the original series is limit-comparable to $\sum \frac{1}{n}$, which diverges, we conclude $\sum \frac{1}{\sqrt{n^2+1}}$ also diverges.*

**Example 10.3.9** *Consider $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^3+5n^2+6n}$. While an integral test is do-able, as is a DCT argument, it seems more natural to note that $\frac{2n+1}{n^3+5n^2+6n} \approx \frac{2n}{n^3} = \frac{2}{n^2}$ for large $n$. We can limit-compare the series to $\sum \frac{2}{n^2}$, or even $\sum \frac{1}{n^2}$. For this example we will do the latter.*

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n^3+5n^2+6n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(2n+1)(n^2)}{n^3+5n^2+6n} = \lim_{n\to\infty} \frac{2n^3+n^2}{n^3+5n^2+6n} \stackrel{\infty/\infty}{\overline{\overline{\text{ALG}}}} \lim_{n\to\infty} \frac{\cancel{n}^3\left(2+\frac{1}{n}\right)}{\cancel{n}^3\left(1+\frac{5}{n}+\frac{6}{n^2}\right)}$$
$$= \frac{2}{1} = 2 \in (0,\infty).$$

*Our original series is limit-comparable to $\sum \frac{1}{n^2}$, which converges, and so therefore must our original series.*

In the conclusive cases where $(a_n)/(b_n) \to L \in \{0, \infty\}$, there is usually DCT argument that would also get us our result. For instance, if we look at $\sum \frac{\ln n}{n}$, for the DCT we can show that $\frac{\ln n}{n} \geq \frac{1}{n}$, and since $\sum \frac{1}{n}$ diverges we know the same is true of $\sum \frac{\ln n}{n}$. But if we instead limit-compare the two series, we get

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty,$$

and since $\sum \frac{1}{n}$ diverges we can safely say that so does $\sum \frac{\ln n}{n}$. (Recall the idea is that $\frac{\ln n}{n}$ is a larger and larger multiple of $\frac{1}{n}$, which is not summable.)

**Example 10.3.10** *Consider* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n} e^{-n}$. *For this we can use either DCT or LCT.*

*For the Direct Comparison Test, we note that since* $\ln n < n$ *for large* $n$*, we have* $\frac{\ln n}{n} e^{-n} < 1 \cdot e^{-n} = \left(\frac{1}{e}\right)^n$*, which is summable (geometric series,* $|r| = \frac{1}{e} < 1$*). Hence so is* $\sum \frac{\ln}{n} e^{-n}$.

*For the Limit Comparison Test, we limit-compare the original series to* $\sum e^{-n}$*, which again converges (see above). So we compute*

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n} e^{-n}}{e^{-n}} = \lim_{n\to\infty} \frac{\ln n}{n} = 0,$$

*which we get from either one l'Hôpital's Rule step or from our hierarchy of functions. Since* $\sum e^{-n}$ *converges, and the limit above is zero, so does* $\sum \frac{\ln n}{n} e^{-n}$ *converge by a special case* $(L = 0)$ *of the LCT. (Recall the idea there is that the given series terms are smaller and smaller multiples of the convergent series* $\sum e^{-n}$*.)*

The limit comparison test can sometimes produce limits which are challenging to compute. Often such examples would be easier with a later technique, as in the next example (which would be more appropriate for a Ratio Test, in Section 10.5, but ultimately such tests rely for their proofs on these more primitive tests.

**Example 10.3.11** *Consider the series* $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n} e^{-n}$.

*This has three functions from our Hierarchy, listed here in the order they appear in the hierarchy:* $\ln n$*,* $n$ *and* $e^n$ *(in a denominator).*

*Now* $\frac{n}{\ln n} e^{-n}$ *may indeed shrink, but not as quickly as* $e^{-n}$*, since* $\frac{n}{\ln n} \to \infty$*. Note that* $e \in (2, 3)$*, so* $1/e \in \left(\frac{1}{3}, \frac{1}{2}\right)$*, so we will compare the original series to* $\sum \left(\frac{1}{2}\right)^n$*, whose terms shrink more slowly than* $\sum e^{-n}$*, so we will see how* $\sum 2^{-n}$ *compares to our original series in the limit.*

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{n}{\ln n} e^{-n}}{2^{-n}} &= \lim_{n\to\infty} \frac{n}{\ln n} \left(\frac{2}{e}\right)^n \\ &= \lim_{n\to\infty} \frac{n}{(e/2)^n} \cdot \frac{1}{\ln n} \overset{0\cdot 0}{=\!=} 0. \end{aligned}$$

*The last computation is based on the heirarchy of functions again, with the polynomial power* $n^1$ *divided by an exponential* $(e/2)^n$*, since* $e/2 > 1$*. This limit can also be arrived at by defining* $y = \frac{n}{\ln n} \left(\frac{2}{e}\right)^n$*, and finding* $\lim_{n\to\infty} \ln y = -\infty$*, implying* $y \to 0$.

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \underbrace{n \ln \frac{2}{e}}_{\substack{\downarrow \\ -\infty}} + \underbrace{\ln n}_{\substack{\downarrow \\ \infty}} - \underbrace{\ln(\ln n)}_{\substack{\downarrow \\ \infty}}$$

*While this looks like an indeterminate case where we need to combine the expressions into one (large) fraction and use l'Hôpital's Rule, in fact if we look at the Hierarchy of Functions, Theorem 10.3.3 (page 824), we see that the first-degree polynomial term* $(\ln\frac{2}{e})n$ *grows much faster in size than* $\ln n$*, and for that matter* $\ln(\ln n)$*, so its effect will dominate as* $n \to \infty$*. Furthermore, the limit above will be* $-\infty$*, since the coefficient of the dominating* $n^1$*-term is* $\ln\frac{2}{e} < 0$*. (*$2 < e \implies \frac{2}{e} < 1 \implies \ln\frac{2}{e} < 0$*, and in fact* $\ln\frac{2}{e} \approx -0.3068528 < 0$*.)*

*However we arrive at the limit, what we have is that*

$$\lim_{n\to\infty} \frac{\frac{n}{\ln n}e^{-n}}{\left(\frac{1}{2}\right)^n} = 0,$$

*and since* $\sum\left(\frac{1}{2}\right)^n$ *converges, so does* $\sum\frac{n}{\ln n}e^{-n}$*.*

While $\sum\frac{n}{\ln n}e^{-n}$ had terms bigger than the geometric series $\sum e^{-n}$, its terms were actually smaller than another geometric series $\sum\left(\frac{1}{2}\right)^n$, whose terms were slightly larger than those of $\sum e^{-n}$.

The argument in the previous example was quite sophisticated, and indeed was not really necessary, once we have the tools of Section 10.5, but much can be done with these more primitive tests. (Also, as mentioned earlier, the more sophisticated tests ultimately rely on DCT for their proofs, so in principle any series using those tests could have convergence or divergence proven by DCT, though it is often much easier to use the computational machinery built into the later tests.

## Exercises

1. Show that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is limit-comparable to $\sum \frac{1}{n}$. Does it converge or diverge?

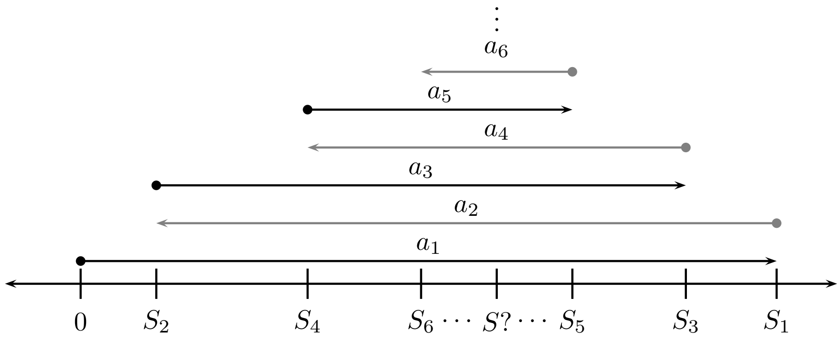


**Figure 10.4:** Illustration of the intuition for the Alternating Series Test. Here $a_1, a_3, a_5, \cdots$ are all positive (represented by black arrows), while $a_2, a_4, a_6, \cdots$ are all negative (represented by gray arrows), $|a_1| > |a_2| > |a_3| > \cdots$, and $|a_n| \to 0$ as $n \to \infty$. This will force $\{S_{2N}\}$ (even partial sums) to be an increasing sequence, and $\{S_{2N-1}\}$ (odd partial sums) to be decreasing. Since these sequences of partial sums are "coming together," meaning their distances from each other $|S_{2N} - S_{2N-1}| = |a_{2N}|$ is shrinking to zero as $N \to \infty$, we see these must converge to some number $S \in \mathbb{R}$. That is the essence of the Alternating Series Test (AST).

## 10.4 Alternating Series and Absolute Convergence

Unlike the previous two sections on positive-term series, in this section we look at series $\sum a_n$ which alternate signs, between positive and negative terms. It turns out that if the terms alternate, and their sizes shrink monotonically to zero, then that is enough for the series to be known to converge. Basically, even if the terms do not shrink to zero quickly, there is a partial cancellation, somewhat like the total cancellation of most terms we saw in telescoping series, though here we are much less likely be able to find a formula for the partial sums.

Some alternating series *rely* on the alternation for convergence, where other alternating series have the terms shrink fast enough that the alternation is not in fact necessary to ensure convergence. The latter series are called *absolutely convergent*, for reasons that will become clear later in this section.

### 10.4.1 Alternating Series Test (AST)

The main result in this section is the following (see Figure 10.4).

**Theorem 10.4.1** *Suppose* $\displaystyle\sum_{n=1}^{\infty} a_n$ *satisfies the following three conditions:*

*(1) The terms of the series alternate signs, i.e.,* $\dfrac{a_{n+1}}{a_n} < 0$ *for all* $n = 1, 2, 3, \cdots$.

*(2)* $|a_1| > |a_2| > |a_3| > \cdots$, *i.e.,* $\{|a_n|\}_{n=1}^{\infty}$ *is a decreasing sequence.*

*(3)* $\displaystyle\lim_{n\to\infty} |a_n| = 0$.

*Rephrased, we suppose the terms of* $\{a_n\}$ *alternate signs and shrink in absolute size monotonically to zero. If this is the case, then* $\sum a_n$ *converges.*

Recall that $\sum a_n$ converges means that the sequence $\{S_N\}_{N=1}^{\infty} = \left\{\sum_{n=1}^{N} a_n\right\}_{N=1}^{\infty}$ of partial sums converges to some finite number $S$, i.e., $S_N \to S \in \mathbb{R}$ as $N \to \infty$. This is partially illustrated in Figure 10.4.

It should also be noted that it is only necessary for the series' terms to show the alternation and shrinking-to-zero (monotonically) hypotheses of the AST "eventually," i.e., for all $n > \mathcal{N}$, some $\mathcal{N} \geq \infty$. In fact it is also enough that $|a_1| \geq |a_2| \geq |a_3| \geq \cdots$ and $|a_n| \to 0$, but usually the stronger inequality ">" is given.

It is also interesing to note that if $|a_n| \not\to 0$, then $a_n \not\to 0$ implying $\sum a_n$ diverges (NTTFD, Section 10.2).

**Example 10.4.1** *Consider the series* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

*If we write out a few terms of this series, we get*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

*Clearly the series' terms alternate signs, and shrink monotonically in absolute value to zero:*

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \lim_{n\to\infty} \frac{|(-1)^{n+1}|}{|n|} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

*If we wish to check for monotonicity of* $|a_n|$, *we can note that the denominator is obviously increasing monotonically, and the numerator is constant, so the fractions which are* $|a_n| = \frac{1}{n}$ *are clearly decreasing (to zero) monotonically. We could also note that if* $f(x) = \frac{1}{x}$, *then* $f'(x) = -1/x^2 < 0$ *for* $x \geq 1$ *(or all* $x \neq 0$*), so* $|a_n| = f(n)$ *is clearly decreasing monotonically.*

*With all this, we know the series converges by the AST.*

Usually a short inspection assures us that the terms in the series shrink in absolute size monotonically to zero, though sometimes sophisticated arguments are required to make this clear.

Note that the convergence would still apply if the signs alternation began with a negative term, as in

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots .$$

In fact the series above would be the additive inverse (negative) of the series in the example above (so they would both converge or both diverge). Furthermore, it should be pointed out that this new series can also be written

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots .$$

One interesting aspect of the convergent alternating series is that we can estimate $|S - S_N|$ easily from an illustration such as Figure 10.4 at the beginning of the section (page 831).

**Theorem 10.4.2** *Suppose* $\sum a_n$ *is an alternating series which satisfies the hypotheses (1)–(3) of the AST, Theorem 10.4.1, page 831. Then*

$$|S_N - S| < |a_{N+1}|. \tag{10.24}$$

Indeed, the distance between $S_N$ and $S$ is always (for such series) less than $|a_{N+1}|$ because by adding $a_{N+1}$ to $S_N$ we "overshoot" $S$ in arriving at $S_{N+1}$.

For a simple application of (10.4.2), suppose we wish to approximate the series $S = \sum \frac{(-1)^{n+1}}{n}$ by taking $S_N$ for $N$ large enough that $|S_N - S| < 0.001$. Then we can use this estimate to find $N$ large enough to be sure $S_N$ is indeed within 0.001 of the full series $S$. We do this by inserting the inequality (10.24) within $|S - S_N| < 0.001$:

$$\begin{aligned} |S - S_N| < |a_{N+1}| \quad & \underbrace{< 0.001}_{\text{inserted}} \\ \implies \frac{1}{N+1} & < 0.001 \\ \implies \frac{1}{0.001} & < N+1 \\ \implies 1000 & < N+1 \\ \implies 999 & < N. \end{aligned}$$

Thus we need $N > 999$, or $N \geq 1000$ to guarantee by (10.24) we have $|S - S_N| < 0.001$.[18]

Other series which we can quickly see converge by the AST follow:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}, \qquad \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right).$$

In fact the last series may warrant a check:

$$f(x) = \sin\frac{1}{x} \implies f'(x) = \left(\cos\frac{1}{x}\right) \cdot \frac{-1}{x^2} < 0$$

for large enough $x$ that $\cos\frac{1}{x} \approx \cos 0 = 1$. Furthermore, $|(-1)^n \sin\frac{1}{n}| \to |\sin 0| = 0$ as $n \to \infty$.

On the other hand, we must be careful to note that alternation alone is not enough to conclude a series converges.

**Example 10.4.2** *Consider* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$.

*Here we have alternation, but*

$$\lim_{n\to\infty} \left|\frac{(-1)^n n}{n+1}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

*and so the series diverges. The last limit can be computed using either algebra (factor $n$ from the numerator and denominator) or l'Hôpital's Rule.*

Similarly, $\sum \frac{(-1)^n n}{\ln n}$ will not coverge, since $|(-1)^n n/\ln n| = n/(\ln n) \to \infty$ as $n \to \infty$. Ultimately it is the NTTFD that lets us conclude divergence, since $|a_n| \to 0 \iff a_n \to 0$.[19]

---

[18] Using methods from the next chapter, we can in fact show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \approx 0.693147181$. To guarantee this kind of accuracy ($\pm 0.000000001 = \pm 10^{-9}$) using (10.24) we would, for this series, need to sum $N = 10^9$ terms, which would require some care and skill even for a computerized computation.

[19] This is not true for limits other than zero; for other limits, we only have $\Longleftarrow$. For instance, $\lim_{n\to\infty} a_n = -3 \implies \lim_{n\to\infty} |a_n| = 3$. In fact $\lim_{n\to\infty} |a_n| = 3$ is even true for the divergent sequence $\{3(-1)^n\}$. However, $|x| = 0 \iff x = 0$, while $|x| = 3 \Longleftarrow x = 3$.

### 10.4.2 Absolutely and Conditionally Convergent Series

Here we point out that there is a convergence which is stronger (more stringent) than our previous definition that $S_N \to S \in \mathbb{R}$. Before arriving at a definition of this stronger convergence criterion, however, we first look more closely at two similar but crucially different alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots, \tag{10.25}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots. \tag{10.26}$$

Note that if we remove the alternation from the first series (10.25), it becomes the harmonic series and therefore diverges. That is not the case with the second series, which if we remove the alternation we get $\sum \frac{1}{n^2}$, which is a $p$-series with $p = 2 > 1$ so it converges.

Putting this another way, the first series *relies* on the alternation to converge; the second series has terms which shrink fast enough that if we did not allow alternation (by removing the factor $(-1)^{n+1}$ or other methods), we still get a finite number for our "infinite series."

The manner in which we detect if the terms shrink fast enough that they do not require alternation is to insert absolute values around each term in the series, which makes each term nonnegative and therefore eliminates the partial cancellation which the AST relied upon for the intuition behind that theorem.

So we note that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \text{ (diverges)},$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots \text{ (converges)}.$$

So somehow the convergence of series (10.26) seems stronger than that of series (10.25), but we must make this more precise, as we do below.

**Definition 10.4.1** *We call a series* $\sum a_n$ **absolutely convergent** *if and only if* $\sum |a_n|$ *converges.*

So the term "absolutely" refers to the absolute values we inserted. However, while "absolutely convergent" seems in this context to mean the series with absolute values inserted does converge, in another context "absolutely convergent" seems to indicate a type of magnified "regular convergence" (in the sense $S_N \to S \in \mathbb{R}$), with "absolutely" as an adjective. In fact, both interpretations are correct, the second one following from the theorem below.

**Theorem 10.4.3** *Suppose* $\sum a_n$ *is absolutely convergent, i.e.,* $\sum |a_n|$ *converges. Then* $\sum a_n$ *also converges, in the sense that its partial sums form a convergent sequence* $\{S_N\}$.

In other words, absolute convergence implies regular convergence:

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges}. \tag{10.27}$$

We will not prove this, since it is more appropriate for a course in real analysis. However it should have the ring of truth.

**Example 10.4.3** *Consider* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

*While this series converges by the AST, we can also prove that it converges absolutely. In fact*

$$\left|\frac{(-1)^n}{n!}\right| = \frac{1}{n!} < \frac{1}{n^2}$$

*for large enough* $n$. *Since* $\sum \frac{1}{n^2}$ *converges, so must* $\sum \frac{1}{n!}$. *Thus* $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ *is absolutely convergent.*

Note that the series in the above example is defined at $n = 0$ while $\sum \frac{1}{n^2}$ is not, but we only need to be sure the "tail ends" of the series are direct-comparable. Note also that once we insert absolute values, we are back to the positive-term series tests, such as the Integral Test, DCT (used above), and LCT.

It should also be pointed out that absolute convergence and convergence are the same for postive-term series. In fact they are also the same for negative-term-only series as well. Only when we know there is some alternation do the two concepts differ.

Finally, if the series has *some* alternation of signs, then $\sum |a_n| \neq \sum a_n$, and in fact $|\sum a_n| < \sum |a_n|$ since (among other reasons) the former allows cancellation and the latter does not.[20]

So we have series that converge absolutely, and series which converge but not absolutely, and series which diverge. For the second type we have another name to identify them more precisely:

**Definition 10.4.2** *If* $\sum a_n$ *converges but not absolutely, we call that series* **conditionally convergent**.

In other words, $\sum a_n$ is conditionally convergent iff $\sum a_n$ converges but $\sum |a_n|$ diverges. In such a case we note that the convergence of the original series must have been due to some alternations of sign,[21] and if we remove the alternation by inserting absolute values around each term, the terms of the series do not shrink fast enough to be summable, so convergence is conditioned on the alternation.

Thus $\sum \frac{(-1)^{n+1}}{n}$ converges (AST), but not absolutely, and is thus conditionally convergent.

So to restate a fact mentioned earlier, any given series is either absolutely convergent, conditionally convergent, or divergent. The union of the first two types is what we simply call convergence ($S_N \to S \in \mathbb{R}$).

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[20] It is always important to distinguish between convergence of a series and what it actually converges to, which are two different questions. Many of the convergence tests do not pretend knowledge of the actual value of the series, though some hints regarding its value may be present in the logic of a given test, or its particular application.

[21] though not necessarily a consistent $+-+-+-\cdots$, since other patterns may similarly account for it, such as $+--+++--+-++-$ or similar.

## 10.5 Ratio and Root Tests

The two tests introduced here compare the behaviors of series to geometric series. For some series, the tests here can conclusively determine if the given series converges absolutely, or diverges. There will be many important series for which the tests apply, and many important series for which the tests are inconclusive (for instance $p$-series).

Note that if a geometric series converges, it does so absolutely, since it is in fact $|r| = |a_{n+1}|/|a_n|$ which is the crucial quantity. In other words, applying the absolute value to each term would not change convergence (though it certainly might change the value of the series).

For the ratio and root tests, we will define a quantity $\rho$ for a given series, and this $\rho$ will mimic $|r|$ from the geometric series. There are two ways we define $\rho$, but they are usually equal (at least for examples found here), though we decline to prove it for this discussion. What we will prove is that they are equal to $|r|$ for a geometric series. Consequently, when $\rho < 1$ (i.e., $\rho \in [0, 1)$) the series will converge; when $\rho > 1$ the series will diverge. When $\rho = 1$ the tests will in fact be inconclusive, as we will show with examples. This tells us that there is some room between the geometric series with $|r| = 1$ (which diverges), and convergent series with $\rho = 1$, as well as divergent series with $\rho = 1$. As hinted at previously, the $p$-series will all have $\rho = 1$.

For the Ratio Test (RAT) we will define

$$\rho_{\text{Ratio}} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Note that for a geometric series $\sum \alpha r^n$, with $\alpha, r \neq 0$, this becomes

$$\rho_{\text{Ratio}} = \lim_{n\to\infty} \left| \frac{\alpha r^{n+1}}{\alpha r^n} \right| = \lim_{n\to\infty} |r| = |r|.$$

Thus $\rho_{\text{Ratio}} = |r|$ for the case of a geometric series.

For the Root Test (ROOT) we instead define

$$\rho_{\text{Root}} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}.$$

For the geometric series this becomes

$$\rho_{\text{Root}} = \lim_{n\to\infty} |\alpha r^n|^{1/n} = \lim_{n\to\infty} |\alpha|^{1/n} \left(|r|^n\right)^{1/n} = |\alpha|^0 |r| = |r|.$$

Rather than distinguish between $\rho_{\text{Ratio}}$ and $\rho_{\text{Root}}$, we will simply refer to both as $\rho$, especially since they are usually the same number.[22] However, it should be clear in all cases which is used from the context.

If we calculate $\rho$ for a series $\sum a_n$, and $\rho < 1$, then we can interpret this to mean $\sum |a_n|$ behaves very much like a geometric series with ratio $\rho$, in the sense that they somehow converge similarly (and absolutely, but probably to different values). The same is true of series $\sum a_n$ where $\rho > 1$, diverging the same way such a geometric series would. Unfortunately the tests below are not comprehensive. For instance, if $\rho = 1$ or does not exist, we can not immediately decide from that fact alone whether or not the series converges (absolutely or otherwise), or diverges. However the tests are quite useful for many and important cases. These tests follow next.

[22] In fact, for most series found in calculus textbooks, $\rho_{\text{Ratio}} = \rho_{\text{Root}}$, so both are just referred to as $\rho$. However, if $\sum a_n$ had infinitely many zero terms, or nearly-zero terms between larger terms (an oscillating sequence of some kind), the Ratio Test might be more problematic than the Root Test, but where both $\rho$'s are defined they will coincide.

**Theorem 10.5.1 Ratio Test (RAT):** *Suppose for a series $\sum a_n$ the limit*

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{10.28}$$

*exists, or is $\infty$. Then*

1. *If $\rho \in [0,1)$, i.e., $\rho < 1$ then $\sum a_n$ converges absolutely.*
2. *If $\rho \in (1,\infty]$, i.e., $\rho > 1$ then $\sum a_n$ diverges.*
3. *If $\rho = 1$ then this test is inconclusive (and some other test must be used).*

**Theorem 10.5.2 Root Test (ROOT):** *Suppose for a series $\sum a_n$ the limit*

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n} \tag{10.29}$$

*exists, or is $\infty$. Then*

1. *If $\rho \in [0,1)$, i.e., $\rho < 1$ then $\sum a_n$ converges absolutely.*
2. *If $\rho \in (1,\infty]$, i.e., $\rho > 1$ then $\sum a_n$ diverges.*
3. *If $\rho = 1$ then this test is inconclusive (and some other test must be used).*

It is somewhat an art to decide which of the two tests (if any) is better suited for calculating $\rho$. While of the two tests the Ratio Test is more often used, there are certainly cases where the Root Test is closer to ideal, as in cases where $|a_n|^{1/n}$ simplify nicely.

**Example 10.5.1** *Consider $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$. Using the Ratio Test, we compute $\rho$. Since computations involving factorials will become more and more important in this and the next chapter, we will write out this particular computation in some extra detail.*

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)n!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} \overset{1/\infty}{=\!=\!=} 0 < 1.\end{aligned}$$

*Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges (absolutely, but that is redundant here since all terms are positive).*

We could have also used our "Hierarchy of Functions" (Theorem 10.3.3, page 824) to just state that $\frac{1}{n!} < \frac{1}{n^2}$ as we once argued, or even $\frac{1}{n!} < \frac{1}{2^n}$, both of which are summable.

**Example 10.5.2** *Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$. We compute $\rho$ for application of the Ratio Test again:*

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{2\cdot 2^n}{2^n} \cdot \frac{n!}{(n+1)\cdot n!} = \lim_{n\to\infty} \frac{2}{n+1} \overset{2/\infty}{=\!=\!=} 0 < 1.\end{aligned}$$

*Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges (absolutely, but again that is redundant here).*

In fact the example above gives us an elegant proof that $\lim_{n\to\infty}\frac{2^n}{n!}=0$. The example shows, by the Ratio Test, that the sereis $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$ converges, and so the contrapositive of the NTTFD subsequently tells us that the terms must shrink to zero, so indeed we conclude that

$$\lim_{n\to\infty}\frac{2^n}{n!}=0.$$

The Ratio Test is clearly quite useful when there is much cancellation in $|a_{n+1}/a_n|$, such as with factorials and some exponential functions.

**Example 10.5.3** *Consider the series* $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^n}$. *A Ratio Test argument would be unwieldy (as the reader is invited to check), so we look instead to the Root Test:*

$$\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left(\frac{2^n}{n^n}\right)^{1/n}=\lim_{n\to\infty}\frac{2}{n}=0<1.$$

*Thus the series converges.*

As noted after the previous example, we then easily get that $2^n/n^n\longrightarrow 0$ as $n\to\infty$, giving another of our orders in our hierarchy of functions. Replacing 2 by any other number will show that $a^n$ grows more slowly than $n^n$. Next we see a proof that $n^n$ grows faster than $n!$.

**Example 10.5.4** *Consider the series* $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

*The $n!$ term seems better suited to the Ratio Test, where the $n^n$ term indicates a Root Test. Since we can more easily deal with a division of powers than a root of a factorial, we will opt for the Ratio Test.*

$$\begin{aligned}\rho&=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}=\lim_{n\to\infty}(n+1)\cdot\frac{n^n}{(n+1)^{n+1}}\\&=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n.\end{aligned}$$

*For this limit, we need some logarithmic arguments. We will let $y=\left(\frac{n}{n+1}\right)^n$ and find $\lim_{n\to\infty}\ln y$:*

$$\begin{aligned}&y=\left(\frac{n}{n+1}\right)^n\\ \Longrightarrow\ &\lim_{n\to\infty}\ln y=\lim_{n\to\infty}n\ln(n/(n+1))\ \overset{\infty\cdot\ln 1}{\underset{\text{ALG}}{=\!=}}\ \lim_{n\to\infty}\frac{\ln n-\ln(n+1)}{n^{-1}}\ \overset{0/0}{\underset{\text{LHR}}{=\!=}}\ \lim_{n\to\infty}\frac{\frac{1}{n}-\frac{1}{n+1}}{-n^{-2}}\\ &\qquad=\lim_{n\to\infty}(-n^2)\left(\frac{n+1-n}{n(n+1)}\right)=\lim_{n\to\infty}\frac{-n^2}{n^2+n}=-1\\ \Longrightarrow\ &\qquad\rho=\lim_{n\to\infty}y=\lim_{n\to\infty}e^{\ln y}=e^{-1}<1.\end{aligned}$$

*Thus $\rho<1$, and the series converges (absolutely, which is yet again redundant here). Note that we knew $\ln n-\ln(n+1)\to 0$ because it is the same as $\ln\frac{n}{n+1}\to\ln 1=0$.*

We can again argue that because $\sum\frac{n!}{n^n}$ converges, we must have

$$\lim_{n\to\infty}\frac{n!}{n^n}=0.$$