Chapter 12

Topics in Analytic Geometry

In this chapter we look at parametric curves and polar coordinates in the plane, vectors, three-dimensional analytic geometry, and spherical and cylindrical coordinates. The parametric curves and polar coordinates we will continue to develop single-variable calculus for, as we will to some extent for vectors, while the other topics are for preparation for multivariable calculus.

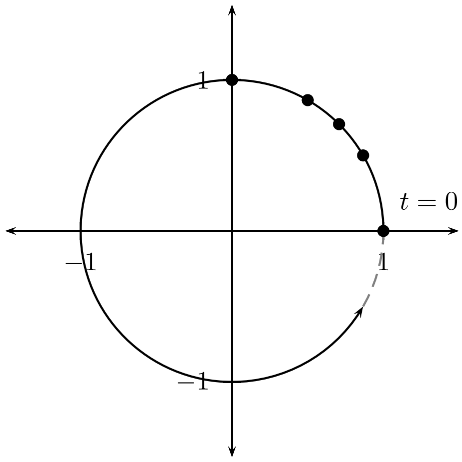
12.1 Parametric Curves

The term **parametric curve** comes from the idea that both the x-coordinate and the y-coordinate of the curve will be functions of a third variable, called the **parameter**. This allows for many more types of curves than those given functionally by $y = f(x)$, and even includes some that would be difficult to give implicitly.

An example of a parametric curve in the plane can be

$$\begin{aligned} x &= \cos t, \\ y &= \sin t. \end{aligned}$$

Thus we follow the x-coordinate as t varies, and separately (if we like) the y-coordinate as t varies. When we graph this for the first time, we might make a chart and graph the points that occur, and attempt to deduce the shape of the graph. Choosing t-values with known sines and cosines, we might produce the following table and graph. Note $\sqrt{3}/2 \approx 0.866$, and $1/\sqrt{2} \approx 0.707$.



t	$x = \cos t$	$y = \sin t$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$
π	-1	0
$7\pi/6$	$-\sqrt{3}/2$	$-1/2$
$5\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$4\pi/3$	$-1/2$	$-\sqrt{3}/2$
$3\pi/2$	0	-1
$5\pi/3$	$1/2$	$-\sqrt{3}/2$
$7\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$11\pi/6$	$\sqrt{3}/2$	$-1/2$

12.2 Polar Coordinates

12.3 Calculus in Polar Coordinates

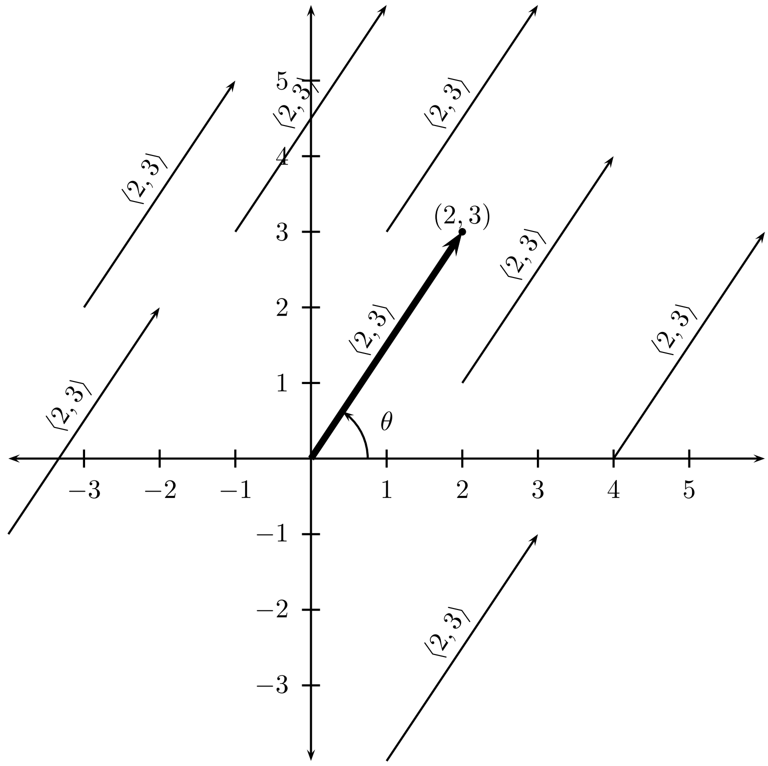


Figure 12.1: For each vector illustrated above, the displacement from the "tail" to the "head" is +2 horizontally, and +3 vertically. Each therefore represent the same (displacement) vector, namely $\langle 2,3\rangle$. If we plot the vector $\langle 2,3\rangle$ in *standard position*, i.e., with its tail at the origin $(0,0)$, we see the head is indeed at $(2,3)$. The length is $\|\langle 2,3\rangle\| = \sqrt{13}$, and the vector points in the direction $\theta = \tan^{-1}\frac{3}{2}$, since it is in the first quadrant.

12.4 Vectors in $\mathbb{R}^2$; Scalar (Dot) Products

The classical definition of a vector is *a quantity with both magnitude and direction.* We will see that there is an intentional ambiguity in this definition.

Another defintion of a vector is *a directed line segment.* This is even more problematic, as it removes exactly the ambiguity that we will see later is crucial.

Both definitions are somewhat geometric. As with derivatives and definite integrals, there is a geometric context in which vectors are easily visualized. However, there are other quantities that are "vector quantities," and there is a purely algebraic definition of a vector.

In this section we will look at vectors "in the plane," particularly the familiar xy-plane, a two-dimensional space. Later in the text we will examine vectors in space, i.e., xyz-space, which is a three-dimensional space.

To introduce the concept of vector, we will use one such example of a vector quantity, which closely mirrors the geometric intuitions. That is the concept of a vector as a *displacement.* As before, a **displacement** can be defined to be a net change in position. Suppose for instance we move from $(0,0)$ to $(2,3)$. This would be a change of +2 in the horizontal position, and +3 in the vertical.

Now suppose instead we move from $(-4,-1)$ to $(-2,2)$. This would also be a change of

+2 in the horizontal and +3 in the vertical directions, repsectively. Both motions represent the same net displacement, which we signify by the **vector** $\langle 2, 3\rangle$. It does not matter what is our initial point, as long as our final point is right 2 and up 3 from our initial point. In all cases it is represented by the same vector $\langle 2, 3\rangle$. See Figure 12.1 at the beginning of this secction.

While it is important that we realize that each of these displacements—of +2 in the horizontal and +3 in the vertical—is considered to be *the same net displacement* and therefore *the same vector*, for many purposes it is best to define a **standard position** for vectors, namely that the tail is fixed to the origin $(0, 0)$. Then the head of the vector $\langle 2, 3\rangle$ would lie at $(2, 3)$.[1]

The standard position of a vector allows us to easily explain these concepts of magnitude and direction. The magnitude is a measure of the vector's size, and the direction is, of course, the direction it points. The length of the vector is given by the notation $\|\langle 2, 3\rangle\|$, and the Pythagorean Theorem gives it to us immediately: $\|\langle 2, 3\rangle\| = \sqrt{2^2 + 3^2} = \sqrt{13}$. More generally,

$$\|\langle a, b\rangle\| = \sqrt{a^2 + b^2}. \tag{12.1}$$

The length of the vector is also known as its **magnitude**, modulus, and sometimes called its absolute value.[2] The direction θ in which the vector points is measured off of the positive x-axis, just as is an angle in standard position. In general,

$$\tan\theta = \frac{a}{b}, , \tag{12.2}$$

and does not have to be in any particular range. Some texts will have $0 \le \theta < 360°$, while others will use $-180° < \theta \le 180°$, but all that is required is that we allow the angles available to describe all possible directions in which a vector can point. Note that we usually decline to define a direction for the **zero vector** $\langle 0, 0\rangle$. Note also that, at the moment we are interested in tnhe geometry of these vectors, no the calculus, so using radian measure for θ is not yet necessary. Some texts defined the *argument* of the vector, $\arg\langle a, b\rangle$ to be some particular angle θ, but we will usually just describe (albeit more verbosely) the angle in question.

When looking at vectors in the plane, it is only necessary to specify the two coordinates a and b of the endpoints (a, b) where the head of vector $\langle a, b\rangle$ points when $\langle a, b\rangle$ is in standard position. Because $a, b \in \mathbb{R}$, we often signify the set of all such possible vectors as

$$\mathbb{R}^2 = \left\{ \langle a, b\rangle \middle| \ a, b \in \mathbb{R} \right\}.$$

Because it takes exactly two numbers to specify to identify the vector, $\mathbb{R}^2$ is called a *two-dimensional space.*

One of the aspects that makes vectors interesting is how they "add" and similarly combine. It is quite intuitive when put simply, and quite interesting when viewed geometrically. When we add two vectors in $\mathbb{R}^2$, we do so as follows:

$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle a_1 + a_2, b_1 + b_2\rangle.$$

The new vector is then called the *resultant* vector, representing the net displacement. As we see from the equation above, to find the net displacement—when we first displace by $\langle a_1, b_1\rangle$, followed by another displacement of $\langle a_2, b_2\rangle$— we look at the total horizontal displacement $a_1 + a_2$ and the total vertical displacement $b_1 + b_2$ to form our new vector. Algebraically this is very simple, but at times we will be interested in a geometric perspective. Geometric vector addition is traditionally described in the following two ways:

[1]This is similar to the "standard position" of an angle θ, which allows much of the analysis—especially of the trigonometric kind—of such objects.

[2]When the length of $\langle a, b\rangle$ is called its it absolute value, the notation usually reflects this as well: $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$.

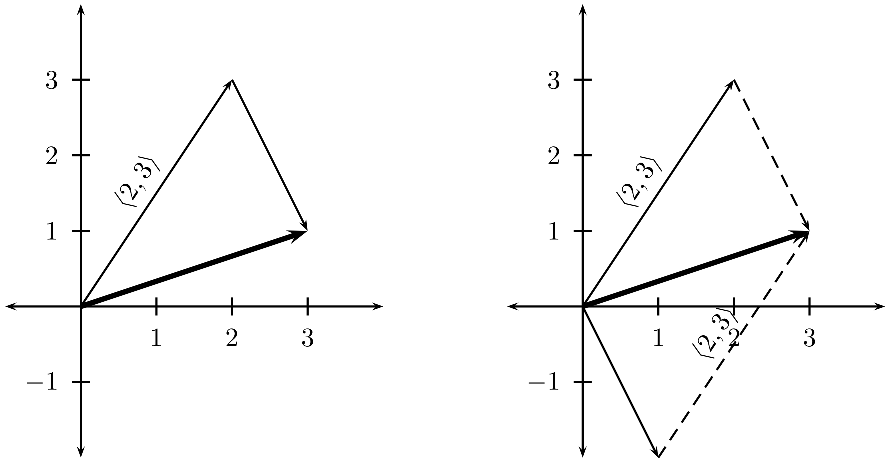
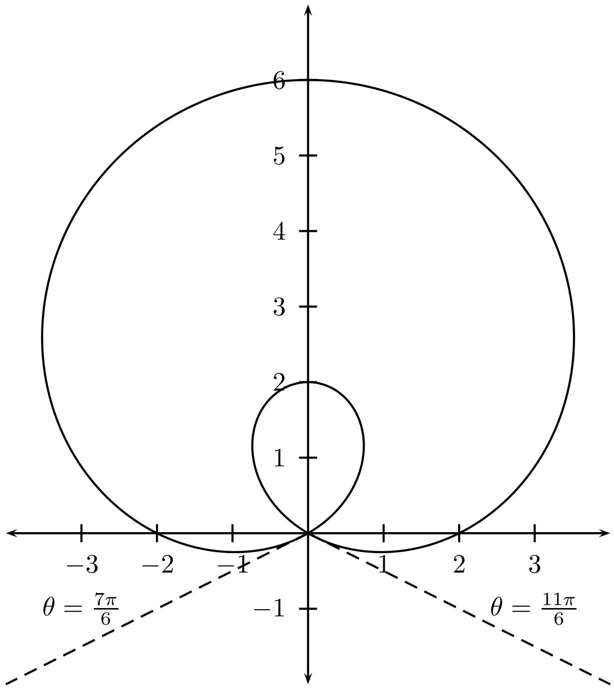


Figure 12.2: Two traditional ways to geometrically describe vector addition. Here we illustrate how these methods predict $\langle 2,3\rangle + \langle 1,-2\rangle = \langle 3,-1\rangle$.

Tail-to-Head: where the origin (tail) of the second vector is placed at the head of the first;

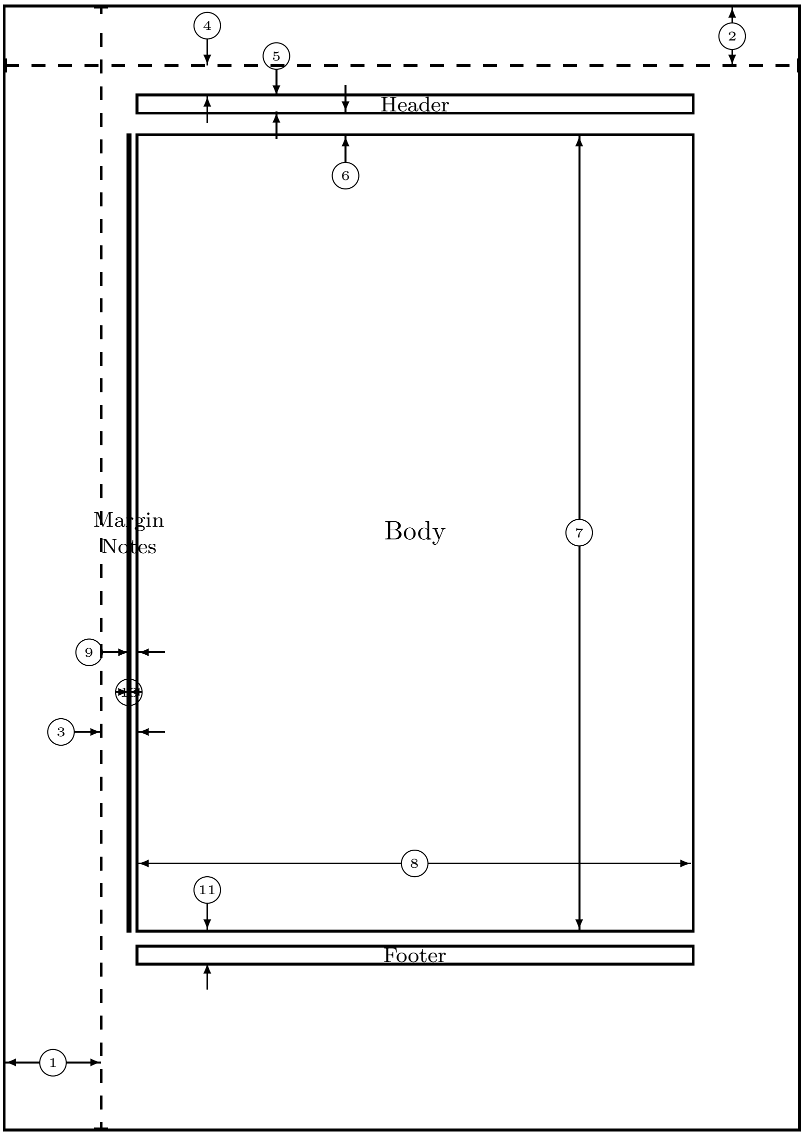
Parallelogram Rule: where we form a parallelogram with the two vectors, in standard position, forming one corner, and the resultant vector coming from that corner to the diagonally opposed corner.


6
5
4
3
2
1
−3
−2
−1
1
2
3
−1
$\theta = \frac{7\pi}{6}$
$\theta = \frac{11\pi}{6}$

12.5 Three-Dimensional Space

12.6 Vectors in $\mathbb{R}^3$; Vector (Cross) Products

12.7 Lines and Planes in $\mathbb{R}^3$



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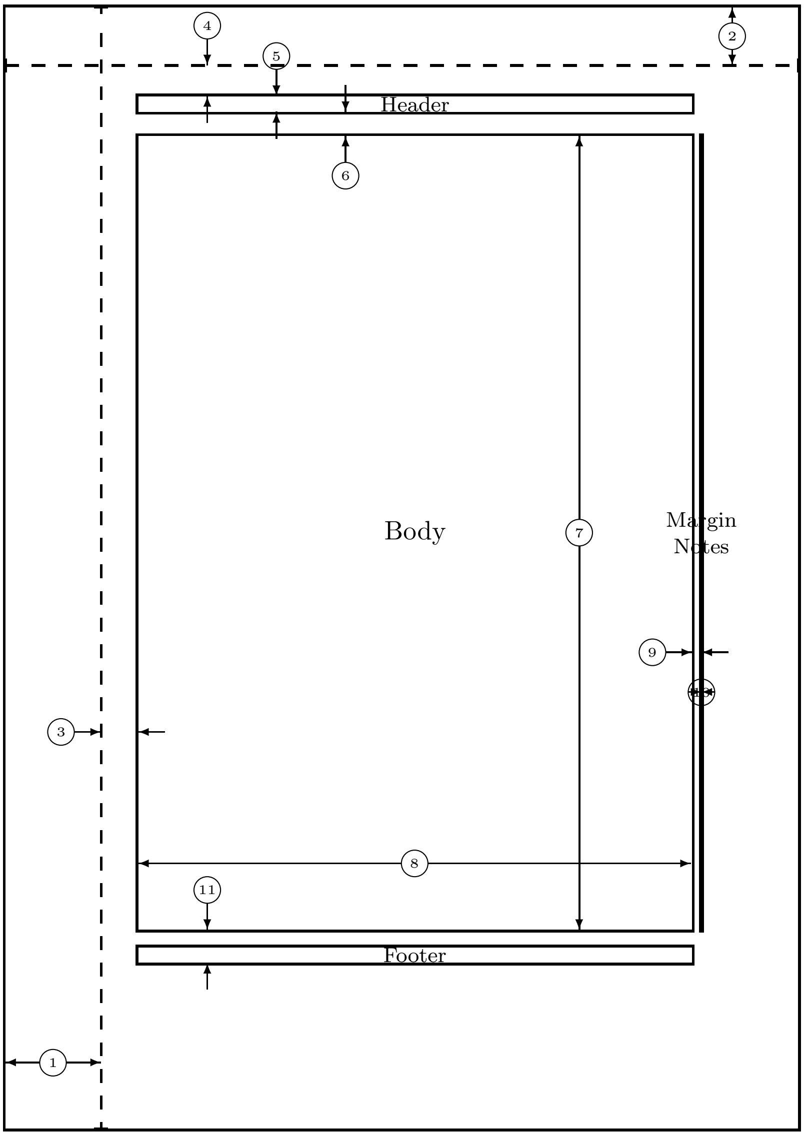

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