Calculus for Students of Mathematics

And Related Disciplines

Preface

Calculus is notable in that any competent mathematician with at least a masters degree, and many with just a strong bachelors, should be fluent enough in the subject to passably teach the courses, since calculus and calculus-descendant studies form such important parts of their training. Perhaps consequently, there are almost as many opinions about how it should be taught as there are people teaching it. A fuzzy and somewhat artificial division into "traditional" and "reform" camps has been the rage for some fifteen years now, though neither seems able to define their own camp very well, let alone the other camp. Textbooks are often sold labeled as traditional or reform, with a growing number giving homage to both. Actually this is not difficult since the camps seem to really disagree mostly on emphasis. When given a choice, professors pick whatever textbook most closely resembles their own philosophy, and make up for the differences using the lectures. When not given a choice, many professors *still* give the nearly the same lectures since again, they understand calculus very thoroughly and have their own ideas about how to best make sense of it to their particular students. If it were not for the scale of such a project, in both writing and dealing with the actual publishing aspects, there would surely be many more—and more diverse—calculus textbooks available to reflect these opinions.

Into this mix I submit this textbook, hoping it will appeal to like minded instructors. It grew out of my own ideas about what was right, and what was lacking in the textbooks I learned and taught from. This text has been in the works conceptually since my own graduate school days, when I was privileged to twice teach summer Calculus I at Purdue University using the very ambitious text by Richard Hunt. When I later, as an assistant professor, found that his second edition would not be published, I searched in vain for an alternative that was of a similar spirit and could find none. Some seven years after teaching those courses at Purdue, and never being totally comfortable with the texts available in the market, I finally began putting my own ideas onto paper.[1] In talking to colleagues over the years, I am led to believe several do share visions similar to my own. To them I offer this as at least a step in their direction. I hope that its format will prove consistent with the goals of those colleagues, and that the text will fill their needs. How well we teach, and how well the students learn calculus are partly functions of our own enthusiasms. I hope that the approach is fresh and energizing to some of my fellow calculus instructors, high school as well as college, who have been looking for a textbook with some of the elements offered here.

Incidentally, the title of this textbook is not meant to exclude students who are in fields of study other than mathematics. Indeed, it is hoped that anyone who pursues calculus for whatever reason will do so as a "student of mathematics." It is not uncommon for a gathering of individuals to contain some who may be considered "students of Shakespeare" but never formally

[1]This is not to say that this textbook is a clone of Richard Hunt's. I only claim that his was my first inspiration, and none of the other available texts seemed to me as inspired with a vision as did his. I have heard Hunt's strategy described as "sneaking in some real analysis." This seems an apt description of his. I do the same, though to some extent I wonder who sneaked the real analysis *out* of the calculus. But my mission is not just to return to that, as the rest of the preface here will explain.

completed a Literature or related major. The phrase here simply means that such individuals care enough about the subject to take personal time to examine it thoughtfully and continuingly, and to become respectably articulate in the subject, at least when among one's peers. On the other hand, one can study the mechanics of, say, calculus without considering its more technical details or its conceptual content. To do so is akin to learning to quote Shakespeare's plays without actually understanding the themes, or knowing the contexts. With calculus as with Shakespeare such lack of understanding can lead to trouble, in the form of embarrassment in the case of Shakespeare, and perhaps more catastrophic consequences in the case of calculus applied to real-world problems. The more of a "student" one is in a particular subject, the better trouble can be avoided and the more the subject can be enjoyed and enriching. Of course this textbook is intended to be thorough for those whose major field of study is Mathematics. However, it is hoped that *Calculus for Students of Mathematics* will inspire each reader—whose study may be any field—to become, for a while if not for a lifetime, a true "student of mathematics."

What is different about this text?

At the risk of appearing trite, I claim that this text is actually *meant to be read.* In my experience many texts are too sparse, and others too concise, in their explanations and it is up to the lecturer to fill in the details or give alternate explanations more fit for student consumption. This text is an attempt to reverse the contemporary roles of mathematics textbook and professor, so that the professor does not have to sprint through the details but can, in good conscience, give the highlights or supplement with his own particular insights, knowing that the students have a complete treatment in the textbook.[2]

Much effort has been made for the text to be self-contained. A reasonably prepared and *dedicated* student should be able to learn enough calculus independently with this text to be able to solve all but perhaps the most challenging problems contained here. The text is naturally more verbose than most, and is peppered with cross references and footnotes. This will be a different style for many students, but one which is worth learning how to read.

Pedagogically Linear Order

This is as opposed to theoretically linear order. I have spent a great deal of thought on the order of topics, and have experimented with various orders extensively with my own classes. I have found that a few simple changes can make profound differences in the rate in which material is absorbed.

While this text is more theoretical than most, it was written with an awareness that there is a momentum to learning. Too many starts and stops in the development can dissipate energy from a calculus class. For that reason, it is sometimes better to show the final, "working" theory than risk bogging down in the preliminary theorems, with or without proofs. For instance, many texts will develop the natural logarithm as a definite integral, show that it works like a logarithm should and therefore must be a logarithm of some kind, and then call some theorem on inverse functions—a topic often painfully developed in its own, barely motivated section—to finally derive the function e^x and its algebraic and calculus properties. I am in good company in deferring the theoretical development—until the reader is well-practiced with both these functions—and then giving the axiomatic theoretical development for completeness. The speed

[2]Perhaps only in mathematics are the professor's lectures traditionally more complete, for the key topics at least, than the text. This is especially true if we include the question and answer dialogues with the students. In contrast, imagine a biology or history professor giving quantitatively more details in lecture than contained in the readings!

in which the computational skills are developed is greater, and the theoretical development is better-appreciated.

I also develop all of the derivative rules in the same chapter (Chapter 4). A reasonable argument can be made that the exponential, logarithmic and arc-trigonometric functions should be introduced later, in between other calculus topics, so students can first further digest the earlier differentiation rules through applications. Though that is a standard pedagogical technique, instead I attempt to exploit the momentum of learning the differentiation rules so that they can be completely dispatched, and then reinforced through use in the chapters on applications. Of course the professor is welcome to break up the material, perhaps to have an exam after the first few sections if it seems appropriate for the particular class.

Similarly, after all the differentiation rules are developed, and a chapter has been devoted to applications of derivatives, I devote two chapters on indefinite integrals before using them in Riemann Sum-motivated applications. The first of these integration chapters exhausts all the functions introduced earlier, in substitution-type settings. The second is my advanced integration technique chapter which builds upon the momentum of the first integration chapter. After these two chapters comes the chapter on Riemann Sums and applications of definite integrals. This approach allows the text to maintain the momentum from the derivative chapters, uninterrupted by Riemann Sums until they can be immediately motivated by the applications, and the student should be able to handle any integral which might arise, since by then the student has accomplished a considerable amount of integration. While this approach is actually less "gentle" for the development of antidifferentiation techniques, it has less stops and starts, and should help the student retain those skills throughout the applications.

Continuity before limits.

One reason I feel comfortable developing a topic completely—without interrupting to allow the reader to "sleep on it"—is that I front-load the text with rigor. Especially in the topics of limits and continuity, my path is perhaps not the quickest through these topics, but rather the path that will give the best hope for a comprehensive understanding. It is coincidentally also the most linear for the theoretical development.

In particular I put continuity before limits, defining both in their own rights, using ε-δ definition. I strongly believe that Calculus loses much rigor when we omit ε-δ (even if students do not always understand these proofs as much as we would like), and that this omission causes much *ad hoc* explanation in the rest of our limit discussions (which can then barely be called "developments"). However, I realize that this is not a real analysis text, and so I only require the student to give ε-δ proofs for the first continuity section where I think it is best motivated (for instance by reference to tolerances), after which theorems ensure we never need to use them again in the exercises. My section on continuity on intervals has a couple of intuitive topological theorems[3] on the images of intervals under continuous functions, from which I can easily state the Intermediate Value Theorem (IVT), and the Extreme Value Theorem (EVT), using the former to give a method for solving polynomial and rational inequalities. I then have several limit sections to take care of all the first semester techniques, including separate sections for vertical and horizontal asymptote phenomena.

Compared to other texts, the extensiveness of this particular chapter is perhaps the most innovative feature of the textbook. It is my sincere hope that it will help solve many of the difficulties associated with teaching these two topics.

Symbolic logic included.

[3]Topological proofs are omitted to avoid the need to define connectedness and compactness.

To help with the rigor and communication of ideas, I include an early introduction to symbolic logic, which I then mix into the prose throughout the rest of the text. This is done for many reasons. First, it adds clarity through precision of the arguments. Second, the symbols naturally illustrate the logical "flow" of the arguments. Finally, it is my hope that this will be a hook for many students who have had difficulty relating abstract mathematics to everyday life, since the symbolic logic arguments have common sense appeal. Learning about logical equivalence is particularly useful in calculus since many theorems are stated in one form, used in another equivalent form, and possibly proved in still another form. Without some logical sophistication, such a discussion can be very confusing for calculus students. In particular, the contrapositive and the difference between implication and equivalence are stressed, as these can be problematic throughout one's college studies and beyond. Of course it is hoped that the discussion of logic will help the dedicated student sharpen his or her own analytical skills in all disciplines, mathematical or otherwise, where logical argument is required.

Studying symbolic logic has several other advantages. For instance, college calculus courses are often populated by a mix of students who had some exposure to calculus in high school, while the rest had none. This often leads to overconfidence in the former group and anxiety in the latter. Beginning with symbolic logic evens the playing field at the start, and sends a clear message to those who had calculus before that college calculus will be different, while giving both the novice and the former high school calculus student an opportunity to build the momentum to study calculus at a college level.

The logic also sets a tone for a generally more abstract text than most. I feel justified in this since, after all, the underlying principles are abstract and understanding these is crucial for proper application. In this spirit I include, for instance, the axiomatic definition of the real numbers (though again, I am aware this is not supposed to be a real analysis text), in order that correct algebraic operations can be discussed in more exalted language. The discussion includes the least upper bound property so that, much later, convergence of sequences and series will not need to be explained in an *ad hoc* manner. Using notation from logic, I give a somewhat different review of algebra and trigonometry than what students may be used to, again to get them thinking about these things from a more sophisticated and hopefully fresher perspective.

Applications.

For applications I stick more to physics examples, and only occasionally inject biological or social scientific examples. I believe physics has the clearest connection to calculus, and offers the best motivations for its study. In fact, I do not use the tangent line slope as my introductory motivation for the derivative, but instead use velocity (vis-à-vis position). I believe velocity is initially more intuitive to more students. The fact that the derivative is graphically the slope of a tangent line is a very convenient device of course, and I exploit it extensively, but too many students walk away uninspired from calculus thinking it is all about tangent lines, and not instead about change (instantaneous *and* cumulative).

Other differences.

Also different is the fact that this text is in black and white, further reinforcing a more abstract spirit. This may be more a matter of taste, but I believe there is a place for such a text and that fancy, four-color illustrations can be distracting from the main themes, not to mention far more expensive to produce, a fact not unnoticed by cash-strapped students.[4]

[4]It has been pointed out that many middle school history textbooks use very sophisticated, ostensibly attractive

The entire textbook is typeset in LaTeXby the author, using the LaTeX book style, with graphics handled by the LaTeX `pstricks` package. Several other LaTeX packages were also used, mainly for modifying the format. No graphics were imported but are all generated using LaTeX code from these packages.

Acknowledgments

First I would like to thank anyone who reads any part of this book. I wrote it for you! Even if you do not read it cover-to-cover, I very much appreciate your interest. And I would like very much to hear back, regardless of your opinion.

For helping to make this work possible, I am most grateful to my wife of ten years, Hung-Chieh Chang (a.k.a. Joy Dougherty), who never showed me any doubt in her mind that this work would eventually be finished, and who put up with the seemingly countless hours I was bonding with several computers to finish this book. Her opinions on things mathematical, pedagogical and artistic were invaluable.

I am also very grateful to Southwestern Oklahoma State University, particularly all in the Department of Mathematics, for their encouragement and support for this project. Few institutions would allow junior faculty so much freedom to undertake a calculus textbook. In particular I was given some release time and was allowed to use photocopied excerpts of the work-in-progress in my calculus courses. This was both risky and expensive for the department, and I sincerely hope my colleagues find that it was worth it.

This text is strongly influenced by James Phelan, my own high school calculus instructor. Though he also taught at a local college, he did not teach specifically to prepare us for the Advanced Placement test—as so many high school instructors are directed to do—but instead taught what *he* thought was a solid course. (With much credit due to him, I passed the AP test with 5/5 anyhow, which gave him much satisfaction and validation!) His mission was to make us literate in mathematics in as many ways as possible while teaching as much calculus as we could absorb as high school seniors. My desire to be a "student of mathematics," that is, to acquire some mathematical sophistication but not necessarily to earn a degree in the field, came first from him.[5]

The professor who, by example, convinced me to become a mathematics major was Shih-Chuan Cheng at Creighton University. The coherence and sophistication of his lecture notes first convinced me of the beauty of mathematics and probably constitute the single greatest influence on the style of this text. Other coursework under John Mordeson and James Carlson at Creighton convinced me that there is a style of learning mathematics which stresses depth and coherence first and breadth second, which is far from the "sink or swim" approach, and from which students can hold their own among their peers from top-tiered schools. In other words, if you have the depth, the breadth can come later.[6]

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designs, and yet middle school students are not likely to be found under the blankets with a flashlight and their history books. Contrast this with the sparsely illustrated Harry Potter books. Granted, fiction can be more "fun," but a good telling of the exploits of Julius Caesar might be more compelling than a chart or graph.

[5]When I was interviewing for my first position as an assistant professor, and was asked about my teaching philosophy, I explained that the students had been reading too much Stephen King mathematics. I wanted them to think they were reading Tolstoy! (A friend later told me, "Yes, and in Russian!") I got the job.

[6]The opposite approach is that students will somehow acquire depth from knowing a breadth of topics. This is arguable, but for most students I personally believe the breadth first approach will breed more confusion and anxiety, and thus be less likely to produce understanding. However, a reasonable argument can be made that student understanding of mathematics more easily goes from particular to general. My counterargument is that students taking 15–18 hours of college courses may not have the time and mental energy to make all the connections (or "synthesize" in education-speak) we hope for. Still, I encourage the reader to have an open mind on the subject, and indeed I sometimes include "drill" exercises in the problems. But in the explanations and examples I attempt to consistently work from a "depth first" philosophy.

I must also thank all who made desktop publishing of mathematics possible. In particular, those in the LaTeX developers community who brought us not only LaTeX but some amazing supplemental packages, particularly `pstricks, multicols, enumerate` and `caption`, and for Adobe for inventing Postscript and PDF standards and, as importantly, keeping them "open" so the TeX community could exploit them for producing publication-quality mathematics with TeX and LaTeX. With these things I was able to produce textbook quality copy to give to my students in class, as well as online, when the text was still in preliminary form, and to present camera-ready copy to a publisher. This ability has been enormously helpful to the development of this text.

Introduction

The discovery of calculus was one of the most important and exciting achievements in the history of intellectual progress. Virtually every field which deals with quantities has benefited from calculus. It allowed Sir Isaac Newton to derive the laws of planetary motion, Albert Einstein to derive relativity, many an economist to model and analyze market variables, and countless other achievements. In particular, it is reasonable to estimate that physics would be centuries behind its present maturity were it not for the availability of calculus.

Despite the technicalities involved in the lofty fields already mentioned, the fundamental principles of calculus are quite accessible, especially now that the subject has been distilled into a more coherent form in the passing of centuries since its initial discoveries. Generations of researchers and authors have refined the presentations to be understandable to motivated college students with a variety of interests. Some larger universities have separate calculus courses specifically for majors in business, agriculture, forestry, and even English, as well as the mainstays of mathematics, engineering and science. Calculus is the marquee mathematical subject for many of these programs of study, particularly science and engineering. Its importance can not be overstated. Even the algebra-trigonometry courses at our institutions have been fashioned largely to groom students for eventual study of calculus.

So what is calculus? The short answer is that it is the field of mathematics which deals with change, both instantaneous and cumulative. Respectively, this means calculus is mainly—but certainly not exclusively—interested in solving the following two problems:

(1) given algebraic relationships among variables, compute their rates of change with respect to each other;

(2) given the rates of change of variables with respect to each other, find the algebraic relationships among the variables.

Indeed the second is simply the first in reverse. For a simple, though abstract example, consider the following questions:

(i) If we know the position s of an object at every time t, can we know the velocity v of the object at every time t?

(ii) If we know the velocity v of an object at every time t, can we know the position s of the object at every time t?

An important concept which will come through in the text, but which should already be intuitive, is that velocity is ultimately just the measure of how the position s is changing with time t, so indeed (i) and (ii) are examples of (1) and (2) above. The answer to (i) is "yes," and the answer to (ii) is "almost." In fact, for (ii) we need some more information, like where the object is (i.e., s) for a particular time t, and then we can usually "pin down" the position s for all time t. So for instance, if we are given a starting point and time, and the velocity at every time from then onwards, then we know where the object is at every time afterwards. In fact, we have even more freedom, for it is enough to know the velocity at all time, and then knowledge of the position at *any* time will determine the position for all time. But knowing velocity at all times is insufficient to finding position; we need one datum on the position to determine position for all time. In contrast, knowing position at all times is sufficient for knowing velocity at all times.

Problems of type (1) are part of the *differential calculus*, also known as calculus of *derivatives*. Problems of type (2) are part of *integral calculus* where, perhaps predictably, we will compute many *antiderivatives*. Problems of this second type tend to be more (sometimes much more) difficult than problems of the first type.

Before we even begin to work in the differential or integral calculus, we will need some preliminaries. We will begin with a chapter on symbolic logic so that we can employ that language throughout the text. Next we exercise that logic on some algebraic preliminaries. Our first preliminaries specific to calculus follow in the concepts of continuity and limits, which together form much of the theoretical foundation of the calculus, and so we will spend considerable effort on these. The bulk of our work is then contained in the chapters on differential calculus and integral calculus. A final major topic is series, which finishes our work here. This last topic will require much of its own foundational development, but is a very important aspect of the classical calculus. Within that study are the answers to questions such as how a calculator can find $\sin 78^\circ$ to ten digits of accuracy (and how we could with pencil and paper as well, though it would require remarkable persistence!), but that is only a very small sample of the usefulness of that theory.

Throughout the text we will see other applications of limits, derivatives, antiderivatives and series, and we will explore as many of those as reasonable for a text of this scope. The development of the analytical tools is our main goal. The student well-versed in the mechanics of those tools will surely (and, it is hoped, easily) find numerous other uses for the methods developed here.

Reading this Book

The main body of the book is organized into Chapters 1, 2, 3, and so on. Chapters are then organized into sections, so for instance Chapter 1 is divided into Sections 1.1, 1.2, 1.3, etc. Most sections correspond to the amount of material a college professor should be able to introduce in a one or two hour lecture, not including time spent answering homework questions. Sections are themselves sometimes divided, so Section 5.2 (i.e., the second section of Chapter 5) may be divided into Subsections 5.2.1, 5.2.2, and so on. This is done to maintain a type of outline format, with each chapter, section and subsection given a title so that it is clear which topic (or subtopic, or sub-subtopic) is being developed at any particular location in the text. Because there are numerous tangential points and clarifications to be made, numbered footnotes—some quite lengthy—are employed extensively so that the regular flow of the text need not be interrupted.[7]

While all this heierarchy and numbering may at first seem excessive, it has become standard practice, and does help to mark where a particular topic is developed. As mentioned in the preface, this book attempts to be a stand-in for an actual professor lecturing at a chalkboard in a calculus class. It is reasonable for students to expect the professor to write the main points on the board in an outline form. Textbook styles, however, differ from the usual lecture-notes outline form, which includes major headings (here chapters), Roman numerals (sections), upper-case Latin letters (subsections), Persian-Arabic numerals (definitions, examples, steps in the general explanation), and so on. But the basic idea of an outline, branching from general to specific, is the same. Of course it is not uncommon for the professor to stand away from the board from time to time and verbally elaborate on various points, or to mention relevant external topics which do not fit neatly into the flow of the outline and indeed may distract from the strictly-defined purposes of the course if included in the written outline. Nonetheless such points can do much to clarify the material, especially by importing relevance to outside topics. The footnotes provide the author of the textbook the same chance, to figuratively step away from the main flow of the text, clarify the discussion and connect it to the rest of the world.

This text also uses numerical labels for equations, theorems, corollaries, definitions, tables and figures. When a particular equation warrants, it is given a sequentially-assigned label based upon the chapter number for easy reference. For instance, Equation (7.23) would be the twenty-third such equation in Chapter 7. Theorems, definitions and figures are similarly numbered. (The exception is that each chapter's footnote labels reset to 1, so that Chapter 1 has footnotes numbered 1, 2, 3, etc., as do Chapters 2, 3, and so on.) These are all standard, formal styles of labeling found in much of the technical literature. This calculus text presents perhaps an ideal opportunity for a first introduction to its extensive use.

The text, while not strictly linear, is mostly cumulative, with new topics constantly referring to earlier topics. Thus the material should be read in the order it is presented, with few exceptions possible (and then best chosen by an instructor).

It may not always be possible for the student to master each topic as it is presented. Noneth-

[7] This is an example of a footnote.

less, it is very important to work as hard as possible to become as proficient as possible in the topics as they are first encountered. This may sometimes require a very slow and deliberate approach to the exercises and explanations. However, it may occasionally be necessary to "table" a confusing topic, in order to return to it after seeing how it fits into the greater scheme. This is natural, and in fact even a topic seemingly "mastered" will undoubtedly benefit from a revisit. Still, strong efforts on all topics will continually pay returns as one's intuition for calculus as a whole is nurtured.

There are numerous comments within the text explaining the importance of the various topics, and relating how difficult particular topics have historically proven to students. Again, all topics are important, but such comments are provided to indicate where particularly strong effort may be required. Comments regarding common mistakes are also common within the text.

Table of Greek Letters

We often use Greek letters in this text, as is standard for technical writing. In calculus, Δ, δ, ε and Σ have particularly special roles, as do θ and ϕ, among others, in trigonometry. We will also use other Greek letters when they are either appropriate stand-ins for their English counterparts, or when we want them to be conspicuous in mathematical expressions. Finally, a reasonably informed student in any technical discipline is expected to eventually know all the letters of the Greek alphabet. For these reasons we include a table of Greek letters below.

A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ε	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

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