Lecture 1: Differential Equations Introduced

January 12, 2007

1 First Introduction

Differential equations are equations which involve derivatives of a function, such as a function $y = y(x)$ (i.e., a function we call y or "y of x"). *The* solution of a differential equation is the set of all functions y which satisfy the equation.

For example, suppose we have the differential equation

$$y'' = 4y. \tag{1}$$

A rather trivial solution is $y = 0$, i.e., the *constant* solution where $y = 0$ for all x. Not so obvious, but easy to verify is that another solution to (1) is $y = e^{2x}$. To check this, we simply compute the left-hand side[1]:

$$\begin{aligned} y &= e^{2x} \\ \Longrightarrow\ y' &= 2e^{2x} \\ \Longrightarrow y'' &= 4e^{2x} \\ \Longrightarrow y'' &= 4y, \qquad \text{Q.E.D.} \end{aligned}$$

We see that $y'' = 4e^{2x} = 4y$ and so (1) is satisfied. A similar check shows that another solution is $y = 5e^{2x}$, and yet another is $y = 100e^{2x}$. In fact, $y = Ae^{2x}$ is a solution for any fixed *constant* $A \in \mathbb{R}$. A constant which appears in the solution (such as A here) and which can be changed without changing the fact that we have a solution is called a <u>*parameter*</u>.

But there are still other solutions. Note that $y = e^{-2x}$ also solves (1). In fact any function of the form $y = Be^{-2x}$, where B is a fixed constant, is a solution.

As we will learn to compute later in the course, ultimately *the* solution of (1) is the set of all functions $y = y(x)$ of the form

$$y = Ae^{2x} + Be^{-2x}, \qquad A, B \in \mathbb{R}. \tag{2}$$

The solution (2) of (1) is then described as a <u>*two-parameter family of curves*</u> (or of <u>*functions*</u> in this case) because we need two parameters to achieve all possible solutions. Notice that $A, B = 0$ gives $y = 0$ as one solution, which is sometimes called the *trivial* solution of (1) because it most obviously solves (1). Not all equations have a trivial solution.

As mentioned, later in the course we will see how to compute (rather than just observe) what are the solutions of equations such as (1).

[1] The symbol "$\Longrightarrow$" is from logic, and means "implies."

2 One-Parameter Curves from Calculus

One sort of differential equation is common in calculus: given a function's derivative, find the function. For example,

$$\begin{array}{rclcrcl} y' &=& \sin x & \Longrightarrow & y &=& -\cos x + C, \\ y' &=& xe^{-x^2} & \Longrightarrow & y &=& -\frac{1}{2}e^{-x^2} + C. \end{array}$$

In general, assuming x lies in some interval on which this all makes sense, we have

$$y' = g(x) \implies y = \int g(x)\,dx = G(x) + C, \tag{3}$$

where $G(x)$ is an antiderivative of $g(x)$ on the relevant interval, and $C \in \mathbb{R}$ is a parameter.

3 Other Examples

So why should we care about differential equations? Below we have a couple of simple examples of where such things are used.

Example 1 *(Population Growth) One of the simplest models of population growth claims that the rate of increase in a population's size (from new members added through cell mitosis, birth rate, etc.) should be proportional to the number of members of the population. So for instance if you have twice the population, new members are formed twice as quickly. If the population at time t is $P(t)$, and P is differentiable (which it usually is not, but population growth models are, after all, approximations) we would have*

$$\frac{dP}{dt} = kP, \tag{4}$$

where k is the proportionality constant. Note that $P = 0$ for all t is a trivial solution.

Equation (4) is called **separable** *since we can manipulate it algebraically to get the P and dP terms on one side, with t and dt terms on the other, in essence separating P and t for* **integration:**[2]

$$\begin{aligned} \frac{dP}{dt} = kP \implies && \frac{dP}{P} &= k\,dt \\ \implies && \int \frac{dP}{P} &= \int k\,dt \\ \implies && \ln|P| + C_1 &= kt + C_2 \\ \implies && \ln|P| &= kt + C_3 \\ \implies && |P| &= e^{kt+C_3} \\ \implies && |P| &= e^{C_3}e^{kt} \\ \implies && P &= \pm e^{C_3}e^{kt} \\ \implies && P &= Ce^{kt}. \end{aligned} \tag{5}$$

The line with the C_1 and C_2 is often skipped with practice, but only when we are confident that we can

[2]Recall that we can only use our rules of integration if the variables all match. For instance,

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$
$$\int y^2\,dy = \frac{1}{3}y^3 + C,$$
$$\int y^2\,dx \text{ can not be computed as it stands!}$$

If y is a function of x, then the last integral exists, but to use the rules we need to write it out in terms of x only, because of the dx (or find dx in terms of y and dy).

combine the arbitrary constants into one, which here is $C_3 = C_2 - C_1$. We do another combination in the last step, realizing that P can not be both the "+" and "−" cases simultaneously, so we take $C \in \mathbb{R}$ to cover both cases. In fact, we note that the $C = 0$ case covers our trivial solution from before. Technically, if $P = 0$ we can not separate (by dividing by P) from the start, but $C = 0$ does give a valid solution. In general, we have to make note of solutions from the start, which might not appear in our parametrizations later. Here we were lucky the parametrization also "picked up" the $P = 0$. When these solutions are not part of the parametrized family, they are called singular.

Note that such a simple population growth model gives us exponential growth. This ignores death rates (though if also proportional, it turns out that including them does not change the exponential nature of the growth), and also ignores crucial external restraints on the population. For instance, a bacterial culture in a petri dish may well grow exponentially, until crowding on the dish and exhaustion of food supply begin to take effect.

The technique for solving separable equations is straightforward (separate and integrate), but following the constants through requires care and practice. We were ultra-careful this time, and will find some shortcuts later, but must be always vigilant to keep constants in their places.

Physics gives us an endless supply of applications of differential equations. Many involve equations of motion. For one-dimensional motion, position is usually written $s = s(t)$. Recall that velocity is then given by $s'(t)$ and acceleration by $s''(t)$. Recall also that the total force applied to an object is given by

$$\text{Force} = \text{mass} \times \text{acceleration}.$$

We will use all this in the following example:

Example 2 *Hooke's Law for a spring with one end fixed says that the force felt by the other end is opposite and proportional to the displacement of that end from equilibrium. In other words, if s is the position of the free end, then*

$$F = -ks.$$

Thus, for example if the free end of the spring is pulled left of its natural resting position (so $s < 0$), the force applied by the spring will be to the right ($F > 0$) and proportional to the amount of the displacement. Similarly $s > 0 \implies F < 0$. Replacing F by ms'' then gives us the differential equation

$$ms'' = -ks,$$

which can be rewritten

$$s'' = -\frac{k}{m}s. \tag{6}$$

With methods we will encounter later, we will be able to derive the solution

$$s = A\cos\left(\sqrt{\frac{k}{m}} \cdot t\right) + B\sin\left(\sqrt{\frac{k}{m}} \cdot t\right), \qquad A, B \in \mathbb{R}. \tag{7}$$

The solution is a two-parameter family, but we notice that all solutions are periodic (which we could easily anticipate), meaning the spring will oscillate, in this case repeating its motion every $2\pi/\sqrt{k/m} = 2\pi\sqrt{m/k}$ units of time elapsed. (The equation (6) ignores dissipative forces.)

4 Implicit Functions, Direction Fields

Recall that in Calculus, nestled between simple derivatives and antiderivatives was the very important topic called implicit differentiation. The approach there was motivated by the fact that there are plenty of interesting curves that are not given by explicit functions $y = f(x)$. Circles, ellipses,

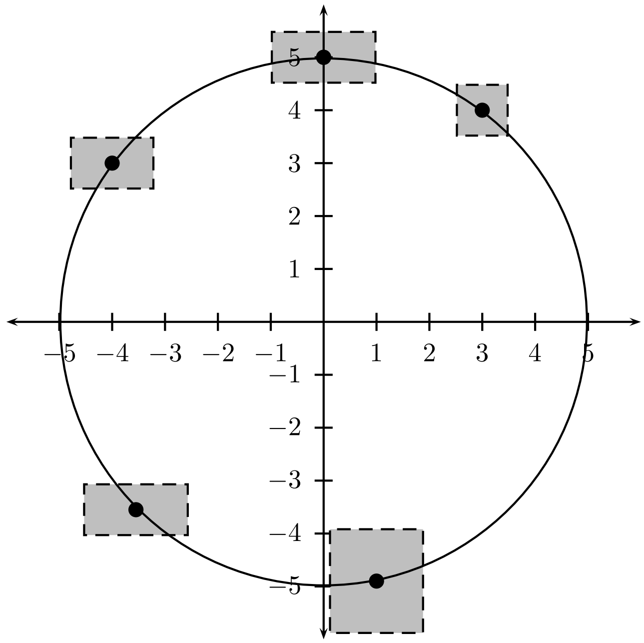

Figure 1: The graph of $x^2+y^2=25$ gives y as a function of x locally except at $(-5,0)$ and $(5,0)$. Some possible open rectangles, centered at various (x_0,y_0) on the graph, and within which y is a function of x are drawn.

some parabolas and the various polar equation curves come to mind. (By Calculus III there are also level curves of functions $f(x,y)$.) Nonetheless, we wanted to find slopes which we still called $\frac{dy}{dx}$. We felt justified because, when we would focus on a particular point on the curve, more times than not the curve *locally* gave y as a function of x. For instance, the circle $x^2+y^2=25$ gives y as a function of x locally except at $(\pm 5,0)$, meaning that we can find open rectangles[3] in which the curve passes the *vertical line test*, making y a function of x there. By vertical line test, we mean any vertical line will pass through the curve at most once, so every x corresponds to at most one y (which is the definition of y being a function of x). So y being locally a function of x near some (x_0,y_0) means that an open box centered at (x_0,y_0) can be found in which the curve passes such a test (without regarding whether the line passes through the curve outside the box). See Figure 1.

Many solutions to differential equations are given by curves (such as circles) which do not represent y as a function of x except locally, as in Figure 1. Recall how we find the slope for such a curve:

$$
\begin{aligned}
& x^2+y^2=25 && (8)\\
\implies & \frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(25) \\
\implies & 2x+2y\frac{dy}{dx}=0 \\
\implies & 2y\frac{dy}{dx}=-2x \\
\implies & \frac{dy}{dx}=-\frac{x}{y}. && (9)
\end{aligned}
$$

[3]not including the boundaries—important to give “wiggle-room” when taking derivatives—but that is a technical point which can be ignored for now

It is very important to point out that the differentiation (applying $\frac{d}{dx}$) is all under the assumption that y is locally a function of x, and so locally both sides of $x^2 + y^2 = 25$ are *the same* functions of x, and therefore have the same derivative. Also, we employ the chain rule to calculate the y-part:

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}.$$

For this case, this is just the chain rule version of the power rule, as in $\frac{d}{dx}(\text{Bob})^2 = 2(\text{Bob})^1 \cdot \frac{d(\text{Bob})}{dx}$, where here $\text{Bob} = \text{Bob}(x)$.

Now let us work backwards. When we had the slope $\frac{dy}{dx}$ as an explicit function of x, i.e., $\frac{dy}{dx} = g(x)$, it was simple to find $y(x) = \int g(x)\,dx = G(x) + C$, but we can not simply integrate when we have $\frac{dy}{dx}$ depending on both x and y, as in (9). However, there is a rather interesting device for visualizing such a case, and it is called, among other things, a *direction field* (and very similar to a *velocity field*, except the former could be one of two opposite velocities). The idea is that since $\frac{dy}{dx}$ gives a slope, it also specifies a direction (really two opposite directions which we will not distinguish for the moment). Thus, at every point where the formula for $\frac{dy}{dx}$ is defined (as a function of both x and y), we can assign a direction which corresponds to the slope $\frac{dy}{dx}$. If we just start at a point in the plane, we can allow the slopes to carry us along, tracing curves with prescribed slope at every point through which we pass. For a very simple case, let us return to the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{10}$$

Next we draw a direction field at every point (well, at every integer point to keep our work finite) using line segments centered at the points and with the same slopes as given by (10) at those coordinates.[4] See Figure 2.

When we are given a general formula for slope

$$\frac{dy}{dx} = f(x, y), \tag{11}$$

we can construct such a direction field and, starting from some desired initial point, follow the field backwards and forwards as we like to trace the curves with prescribed slope along the path. There are complications, since f might not be defined on the whole plane. We will deal with how far we can follow such curves later in the course.

Actually we are lucky in that we can analytically solve (10) because it is separable:

$$\begin{aligned}
\frac{dy}{dx} = -\frac{x}{y} \implies & \qquad y\,dy = -x\,dx \\
\implies & \quad \int y\,dy = \int x\,dx \\
\implies & \qquad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1 \\
\implies & \qquad y^2 = -x^2 + 2C_1 \\
\implies & \quad x^2 + y^2 = C.
\end{aligned}$$

Thus the curves which give rise to (10) are indeed circles. To summarize what we have done so far regarding (10), note that we

[4]Where $y = 0$ the slope here is undefined, but here we gave the vertical direction, since slopes $|dy/dx| \to \infty$, i.e., slopes become vertical, as $y \to 0$. Sometimes we simply ignore those points and do not try to complete a curve where the slope is undefined. It depends upon what kind of curves we are looking for. If we were looking for functions we would leave the field undefined there.

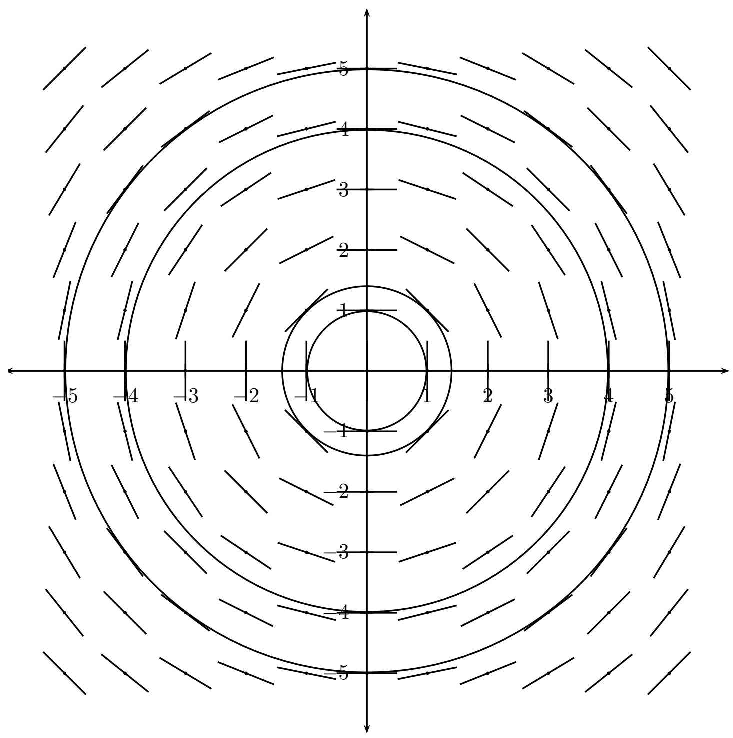

Figure 2: Direction field with some solutions for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. Solutions (also called integral curves) are circles centered at $(0, 0)$.

- recalled implicit functions and implicit differentiation, as in $x^2 + y^2 = 25 \implies dy/dx = -x/y$ (note how the 25 "disappeared" in the differentiation; any constant in its place would as well);
- drew the direction field for $dy/dx = -x/y$, and some curves (which were all circles, though we did not actually derive that fact at that point) with the prescribed slope;
- conversely showed how $dy/dx = -x/y$ leads to circles $x^2 + y^2 = C$ by integration.

This we recall only to show how these things are connected. When we have Equation (11), i.e., $dy/dx = f(x, y)$, those curves which satisfy this slope equation are called *integral curves*.

Consider another direction field, this time for the equation

$$\frac{dy}{dx} = \frac{x}{y}. \tag{12}$$

It is sketched in Figure 3. A pattern for the types of integral curves we can expect there becomes quickly apparent with a glance at the direction field sketched in the figure.

Homework 1-A

1. Verify that (2) satisfies (1), i.e., that $y = Ae^{2x} + Be^{-2x}$ is always a solution to $y'' = 4y$.
2. Verify that (5) is a solution to (4).
3. Verify that (7) is a solution to (6). (It is important to remember k, m are constants, and so therefore is $\sqrt{k/m}$.)
4. Solve (12) by separating the variables and integrating. What are the solutions? Is that reasonable given the drawing of the direction field in Figure 3?

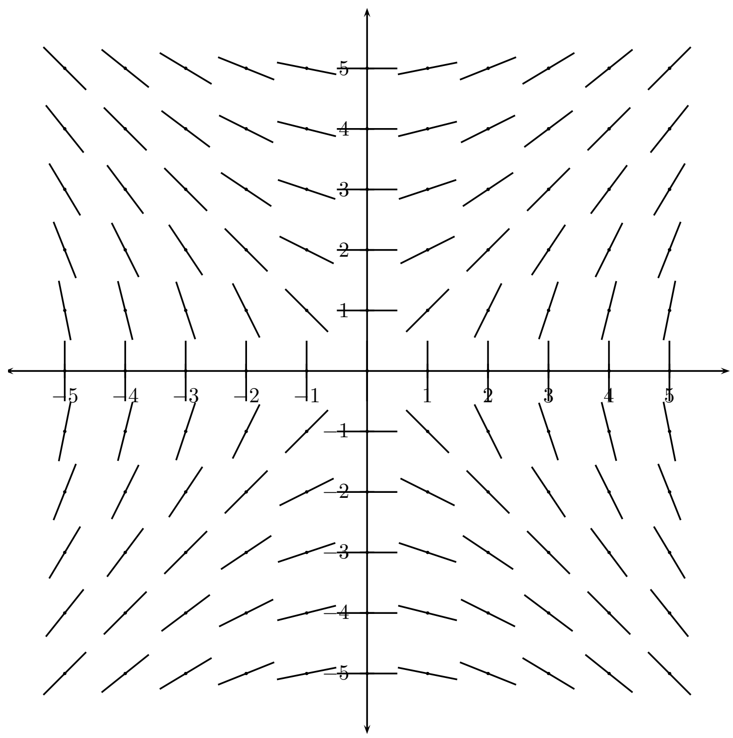

Figure 3: Direction field for $\frac{dy}{dx} = \frac{x}{y}$ suggesting possible shapes for the integral curves.