Lecture 2: Separable Ordinary Differential Equations I

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1 Some Terminology: ODE's, PDE's, IVP's

The differential equations we have looked at so far are called **ordinary differential equations**, or **ODE's**, because they involve ordinary derivatives (of any order) like $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{dP}{dt}$ and so on. Another class of differential equations which we are not likely to have time for in this semester is the **partial differential equations**, or **PDE's.** Those involve partial derivatives. For instance, Laplace's equation (also known as the equilibrium heat equation) is given by

$$u_{xx} + u_{yy} = 0,$$

where $u = u(x, y)$. Heat, wave and electromagnetic theory equations are the most well-known PDE's, but they are ubiquitous in science, engineering and economics. When we have a function (such as price) of more than one variable the derivatives are necessarily partial derivatives. The techniques for solving such equations are many, varied and different from those for ODE's. They are reserved for the course Differential Equations II, which runs when there is a demand.

When we solve an ODE, we are likely to be left with parameters, and therefore a family of solution curves. This family of curves, written with parameters, is often called the **general solution** of the ODE. To determine a specific curve we might need, we usually specify some data which pins down a (hopefully unique) curve satisfying the equation and the data. An equation, together with data to determine a specific curve, is called an **initial value problem**, or an **IVP**. Of course we need to know how to solve the underlying ODE before we can solve any IVP's.[1] Nonetheless, we can look at a couple of IVP's to see what we are in for there.

Example 1 *Let us solve the following IVP:*

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}, \\ y(1/2) = 7. \end{cases} \tag{1}$$

The first part is the ODE, and the second is the initial data (technically "datum" for this case). Now clearly this is separable, so we continue as before:[2]

$$\begin{aligned} \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \implies& \quad dy = \frac{dx}{\sqrt{1-x^2}} \\ \implies& \int dy = \int \frac{dx}{\sqrt{1-x^2}} \\ \implies& \quad y = \sin^{-1} x + C. \end{aligned}$$

Now that the ODE is solved, we need only find the exact curve by finding the C which corresponds to the initial data. For that we just "plug in" $(x, y) = (1/2, 7)$:

$$7 = \sin^{-1}(1/2) + C = \frac{\pi}{6} + C \implies 7 - \frac{\pi}{6} = C.$$

[1] This is where the terminology gets fun.

[2] Or we could simply write $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,dx \implies y = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C.$

Thus the solution to the IVP is

$$y = \sin^{-1} x + \left(7 - \frac{\pi}{6}\right). \tag{2}$$

Example 2 *Consider the IVP*

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} & \textit{(the ODE)} \\ y(8) = -6. & \textit{(initial data)} \end{cases} \tag{3}$$

As we saw in Lecture 1, the general solution to the ODE is $x^2 + y^2 = C$. *(By now you should be able to quickly visualize how we got that.) Now using the data point* $(8, -6)$ *we get*

$$(8)^2 + (-6)^2 = C \implies 64 + 36 = C \implies C = 100.$$

Thus the circle $x^2 + y^2 = 100$ *solves the IVP. Or does it? To be sure, the circle does not give us* $y = y(x)$ *(i.e., does not give* y *as a function of* x*). If we want* y *as a function of* x*, we have to solve for it. Since* $y^2 = 100 - x^2$*, we have*

$$y = \sqrt{100 - x^2} \qquad \textit{or} \qquad y = -\sqrt{100 - x^2}.$$

So which is it? The first is the upper semicircle, and the second is the lower. The data point $(8, -6)$ *is on the lower semicircle, so if we want a* **function** *(and not just a* **curve***), we have to conclude the solution to be*

$$y = -\sqrt{100 - x^2}. \tag{4}$$

Sometimes we would rather have the whole curve, and other times it is best to have a function. Usually the latter is preferable whenever possible.

Sometimes we need more than one piece of data. Since we may have more than one parameter in the general solution, from linear algebra we know that two equations are required to find two unknowns, and it is often the case here too. Next we have just such an example.

Example 3 *Solve the IVP*

$$\begin{cases} y'' = -4y \\ y(\pi/8) = 2\sqrt{2} \\ y'(\pi/8) = 6\sqrt{2}. \end{cases} \tag{5}$$

Later in the course we will see how to solve the ODE, and find that the general solution is $y = A \sin 2x + B \cos 2x$. *(You should be able to see readily that* $y = \sin 2x$ *and* $y = \cos 2x$ *are both solutions of the ODE, i.e., the top of* (5)*, and be able to show, if asked, that* $y = A \sin 2x + B \cos 2x$ *is* a *solution if not* **the** *solution to the ODE.) Now we enter the data:*

$$\begin{aligned} y &= A\sin 2x + B\cos 2x, & y\left(\frac{\pi}{8}\right) &= 2\sqrt{2} \implies & A\sin\left(\frac{\pi}{4}\right) + B\cos\left(\frac{\pi}{4}\right) &= 2\sqrt{2} \\ y' &= 2A\cos 2x - 2B\sin 2x, & y'\left(\frac{\pi}{8}\right) &= 6\sqrt{2} \implies & 2A\cos\left(\frac{\pi}{4}\right) - 2B\sin\left(\frac{\pi}{4}\right) &= 6\sqrt{2}. \end{aligned}$$

This gives us a system of two equations in two unknowns, A *and* B*:*

$$\begin{array}{rcrcl} A\cdot\frac{1}{\sqrt{2}} & + & B\cdot\frac{1}{\sqrt{2}} & = & 2\sqrt{2} \\ 2A\cdot\frac{1}{\sqrt{2}} & - & 2B\cdot\frac{1}{\sqrt{2}} & = & 6\sqrt{2} \end{array} \quad \underset{\textit{(simplifying each)}}{\implies} \quad \begin{array}{rcrcl} A & + & B & = & 4 \\ A & - & B & = & 6. \end{array}$$

Adding the equations gives $2A = 10$*, or* $A = 5$*. Subtracting the equations gives* $2B = -2$*, or* $B = -1$*. Thus the solution to the IVP is*

$$y = 5\sin 2x - \cos 2x. \tag{6}$$

2 Separable Equations Defined; Constant Solutions

A first-order ODE in which the y, dy and x, dx terms can be algebraically reorganized to opposite sides of the equal sign is called separable. We have encountered and solved some of these already. Zill's precise and classical definition is consistent with that idea (Zill, page 50):

> A first-order differential equation of the form
> $$\frac{dy}{dx} = g(x)h(y) \tag{7}$$
> is said to be **separable** or to have **separable variables**.

If we take his definition, we see we can get

$$\frac{dy}{h(y)} = g(x)\,dx \qquad \Longrightarrow \qquad \int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

If we can find formulas for the antiderivatives above, we can then write the solution

$$P(y) = G(x) + C, \tag{8}$$

where $P' = \frac{1}{h}$ and $G' = g$. It will be a one-parameter family of curves, and if we are lucky, we can solve (8) for y. When we are faced with an IVP, with the ODE part separable, we need to plug the data into (8) to find C to get a curve, and possibly solve for y to get an actual function.

In some sense these are conceptually the simplest differential equations to solve. In fact, they fit nicely into any Calculus course when time permits. However, these equations can certainly tax one's integration skills, and solving for y may require some careful algebra. Furthermore, there are often constant solutions $y = k$ that may or may not be contained in the parametrized family of curves, as we will see in the remaining examples.

Example 4 *Solve the following differential equation:*

$$\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{y}. \tag{9}$$

Solution: First note that $y = \pm 1$ are both constant solutions to (9). Indeed, for $y = 1$ or $y = -1$, both the left-hand and right-hand sides will be zero. For other solutions we see that the right-hand side is a separable product as in Zill's definition (7), and we can solve the equation as follows:

$$\begin{aligned}
\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{y} \quad &\Longrightarrow \quad \frac{y}{\sqrt{1-y^2}}\,dy = x\,dx \\
&\Longrightarrow \quad \int \frac{y}{\sqrt{1-y^2}}\,dy = \int x\,dx \\
&\Longrightarrow \quad -\sqrt{1-y^2} = \frac{1}{2}x^2 + C.
\end{aligned}$$

Note that the integral on the left calls for Calc I-type substitution:

$$\left.\begin{array}{rrcl}
 & u & = & 1-y^2 \\
\Longrightarrow & du & = & -2y\,dy \\
\Longrightarrow & \frac{du}{-2} & = & y\,dy
\end{array}\right\} \Longrightarrow \quad \int \frac{y\,dy}{\sqrt{1-y^2}} = \int u^{-1/2}\frac{du}{-2}$$

$$\begin{aligned}
&= \frac{\left(\frac{u^{1/2}}{1/2}\right)}{-2} + C_1 \\
&= \frac{2u^{1/2}}{-2} + C_1 \\
&= -u^{1/2} + C_1 \\
&= -\sqrt{1-y^2} + C_1.
\end{aligned}$$

Again, putting these together gives us

$$-\sqrt{1-y^2}=\frac{1}{2}x^2+C. \tag{10}$$

If we now wish to solve for y, that is also possible (note "multiplying out" the square is optional):

$$\begin{aligned}
-\sqrt{1-y^2}=\frac{1}{2}x^2+C \implies\quad & 1-y^2=\left(\frac{1}{2}x^2+C\right)^2\\
\implies\quad & 1-y^2=\frac{1}{4}x^4+Cx^2+C^2\\
\implies\quad & -y^2=\frac{1}{4}x^4+Cx^2+C^2-1\\
\implies\quad & y^2=-\frac{1}{4}x^4-Cx^2-C^2+1\\
\implies\quad & y=\pm\sqrt{-\frac{1}{4}x^4-Cx^2-C^2+1}\ \overset{or}{=}\ \pm\sqrt{1-\left(\frac{1}{2}x^2+C\right)^2}.
\end{aligned}$$

In the first line, we squared both sides. The rest should be clear. However, recall that in squaring both sides we can introduce extraneous solutions, because we lose some precision in our information about the quantities involved.[3] *Whether a particular solution works will rely to some extent on C (note, for instance, that we can not have $C>0$ because then we would have LHS ≤ 0 and RHS >0 in* (10)*—see why?).*

Also notice where the parameter finds itself in the final form of this solution (twice in the first form). Thus we have to be careful not to just follow the Calculus I and II mentality of "slapping a $+C$ at the end" of every problem.

Also note that if this ODE were part of an IVP,

- *it would surely be simpler to find C in the original form* (10) *of the general solution; and*
- *in the final solution, we can only have one case from the $+/-$ if we are to have a function, which is the whole point of solving for y.*

Further, note that the cases $y=\pm 1$ are not possible to obtain by clever choices of C; they are outside the one-parameter family of curves. In such a case they are called (by Zill, p. 7) **singular solutions.** *In other examples we will have nonsingular constant solutions.*[4]

Finally. note that the solution to (9) *is in $y=-1$, $y=1$ and* (10)*, or equivalently,*

$$y=1,\qquad y=-1,\qquad y=\pm\sqrt{1-\left(\frac{1}{2}x^2+C\right)^2}. \tag{11}$$

The previous example had moderately simple calculus and algebra, though both did require care. Consider next the following.

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[3]A simple example of how we lose information when we square both sides is the following:

$$x=-5\qquad\implies\qquad x^2=25\qquad\implies\qquad x=-5,5.$$

In particular we lose some information about the signs of the quantities involved whenever we square both sides, and can be in danger of admitting **extraneous** solutions such as the $x=5$ case above.

[4]In mathematics, **singular** usually refers to something enigmatic (but still important), where the simpler mathematical analysis breaks down in some way. However the precise meaning changes with context. For a function, a singular point might be where we "divide by zero," as in the point $x=0$ in the function $f(x)=1/x$, in that case resulting in a vertical asymptote, or the point $x=1$ in the function $g(x)=(x^2-1)/(x-1)$, resulting in a removable discontinuity. In matrix theory, a singular matrix is a square matrix $A_{n\times n}$ where A^{-1} does not exist, which is also one in which $\det(A)=0$, since $A^{-1}=\frac{1}{\det A}[\operatorname{adj}(A)]$ (see any linear algebra book), where $\operatorname{adj}(A)=[\operatorname{cof}(A)]^{\mathrm{T}}$, i.e., the transpose of the matrix of cofactors of A, so we are again dividing by zero if $\det(A)=0$. In our present context, there may be constant solutions to an ODE which are outside the parametrized family, and these solutions are called *singular.* They are "different" from what we get from simply varying the parameters, such as C here; there is a definite "leap" to these solutions, whereas many of the curves in the parametrized families can be "morphed" from one to another by continuously varying C through some interval. We will see more of this phenomenon later.

Example 5 *Sometimes differential equations are written without being first solved for $\frac{dy}{dx}$. Indeed, sometimes the dx and dy are both written as multiplicative factors at the outset. (While this seems an unnecessary complication here, it will be important later.) Consider*

$$dy - xy\ln x\ln y\,dx = 0. \tag{12}$$

It is almost always instructive to examine the equation as solved for $\frac{dy}{dx}$ before proceeding farther:

$$\begin{aligned} dy &= xy\ln x\ln y\,dx \\ \Longrightarrow \frac{dy}{dx} &= xy\ln x\ln y. \end{aligned}$$

At this point we look for constant solutions as before. Again, these are solutions of the form $y = k$ for which the RHS is zero, and since they are constant solutions the LHS will also be zero. It appears that $y = 0$ will make the y-factor on the left zero, but then the $\ln y$ term is undefined, so $y = 0$ is not a valid solution. However, when $\ln y = 0$, i.e., when $y = 1$, we do have a solution to the ODE (but only for $x > 0$, as the reader should check). Next we separate the y and x terms and integrate:

$$\begin{aligned} \frac{dy}{dx} = xy\ln x\ln y \Longrightarrow \quad & \frac{dy}{y\ln y} = x\ln x\,dx \\ \Longrightarrow & \int\frac{dy}{y\ln y} = \int x\ln x\,dx. \end{aligned}$$

The first integral yields to substitution, with $u = \ln y$ implying $du = \frac{1}{y}\,dy$:

$$\int\frac{dy}{y\ln y} = \int\frac{1}{u}\,du = \ln|u| + C_1 = \ln|\ln y| + C_1.$$

The x-integral will require integration by parts, with

$$\begin{array}{rcl|rl} u &=& \ln x & dv &= x\,dx \\ du &=& \frac{1}{x}\,dx & v &= \frac{1}{2}x^2 \end{array}$$

giving us

$$\begin{aligned} \int x\ln x\,dx &= uv - \int v\,du \\ &= \frac{1}{2}x^2\ln x - \int\frac{1}{2}x\,dx \\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C_2 = \frac{1}{4}x^2(2\ln x - 1) + C_2. \end{aligned}$$

Putting these together gives us $\ln|\ln y| + C_1 = \frac{1}{4}x^2(2\ln x - 1) + C_2$, or

$$\ln|\ln y| = \frac{1}{4}x^2(2\ln x - 1) + C_3. \tag{13}$$

Again we seemed to have solved the ODE (no more derivatives present!), if we include $y = 1$ which is undefined in (13), but more can be done to solve for y itself. We can take the natural exponential of both sides to then get

$$|\ln y| = e^{\left[\frac{1}{4}x^2(2\ln x-1)+C_3\right]} = e^{C_3}e^{\left[\frac{1}{4}x^2(2\ln x-1)\right]} = C_4e^{\left[\frac{1}{4}x^2(2\ln x-1)\right]}.$$

This then gives

$$\ln y = \pm C_4e^{\left[\frac{1}{4}x^2(2\ln x-1)\right]},$$

and it can't be both cases (+ and −) simultaneously, so we might as well write

$$\ln y = C_5 e^{\left[\frac{1}{4}x^2(2\ln x - 1)\right]}.$$

Finally, we can take the exponential of both sides again to get $y = e^{Ce^{\left[\frac{1}{4}x^2(2\ln x-1)\right]}}$*, or*

$$y = \exp\left[C\exp\left(\frac{1}{4}x^2(2\ln x - 1)\right)\right]. \tag{14}$$

Once again we see that the calculus was moderately difficult, but then even the algebra became rather lengthy (though no particular part of that was difficult). We really should check to see if all constants C *here work (since* $C = \exp(\pm C_4) \neq 0$*), but we will see below that for this case* $C = 0$ *works as well in the original. In fact, though the* $C = 0$ *case technically disappears as we move from* (13) *to* (14)*,* $C = 0$ *would have been exactly the valid case* $y = 1$*, so it "reappears" in* (14) *if we allow* C *to vary more than its definition* ($C = \pm C_4$) *would normally allow. Otherwise we also would have had the full solution before, if we left the two cases separate, though it is perhaps not as presentable as* (14):

$$y = 1 \qquad \textit{or} \qquad \ln|\ln y| = \frac{1}{4}x^2(2\ln x - 1) + C_3. \tag{15}$$

We can see more of this phenomenon of easy, constant solutions (which may or may not be in the parametrized family) again in the following example.

Example 6 *Consider the equation*

$$\frac{dy}{dx} = \frac{y^2 - 1}{y}\sin 2x. \tag{16}$$

Note that $y = 1$ *and* $y = -1$ *are constant solutions of this ODE. For any other solutions we can divide (while for these two we would be dividing by zero!), and get*

$$\begin{aligned}
\frac{y\,dy}{y^2-1} = \sin 2x\,dx \implies &\quad \int \frac{y\,dy}{y^2-1} = \int \sin 2x\,dx \\
\implies &\quad \frac{1}{2}\ln|y^2-1| = -\frac{1}{2}\cos 2x + C_1 \\
\implies &\quad \ln|y^2-1| = -\cos 2x + C_2.
\end{aligned}$$

We can thus give the general solution as

$$y = 1, \quad y = -1, \quad \textit{or} \quad \ln|y^2 - 1| = -\cos 2x + C_2. \tag{17}$$

Now in the third solution one can solve for y *algebraically, and see if these two constant solutions are contained in the new parametrized family if we allow the (new) parameter to range over more than we normally would, as happened in the previous example. The details are left as an exercise.*

Homework 2-A

1. Find some constant solutions for the differential equation

$$\frac{dy}{dx} = (y-5)(\sin y)\exp(x^3).$$

(Please do not try to find the general solution for this ODE!)

2. What if we include a factor of $\ln y$, as in

$$\frac{dy}{dx} = (y-5)(\sin y)(\ln y)\exp(x^3)?$$

How does this change the constant solutions?

3. As in Example 5, see if the constant solutions $y = \pm 1$ for Example 6 "reappear" with the new parameter's extended range if we solve for y in the implicit equation

$$\ln|y^2 - 1| = -\cos 2x + C_2.$$

4. Solve (including all constant solutions)

$$xy^3\,dx + (y+1)e^{-x}\,dy = 0.$$

Hint: First solve for $\frac{dy}{dx}$. Also be slightly clever with algebra. The hard part of the answer (so you can check) is:

$$\frac{1}{y} + \frac{1}{2y^2} = e^x(x-1) + C_1.$$

(This problem comes from the classical text of Rainville and Bedient, 1981 edition.)

5. Solve for y in the previous problem. Hint: multiply both sides by y^2, move all terms to one side of the equation with zero on the other, and use the quadratic formula but with "x" played by y here, and proper care taken to identify the "a, b, c" terms:

$$a \neq 0,\ ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$