Lecture 4: Exact ODE's

January 24, 2007

Exact equations are first-order ODE's of a particular form, and whose methods of solutions rely upon basic facts concerning partial derivatives, exact differentials and level curves. These are all Calc III topics, which we review before introducing the method. The basic idea is that we begin with a first-order ODE which we write

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1}$$

We call the ODE (1) **exact** if the LHS can be written as df, which by definition is given by[1]

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy. \tag{2}$$

In other words, if we have the following situation:

$$\underbrace{M(x,y)dx}_{f_x(x,y)dx} + \underbrace{N(x,y)dy}_{f_y(x,y)dy} = 0. \tag{3}$$

If so, then the ODE is simply $df = 0$, the solution of which is the one-parameter family of curves (giving y as a function of x implicitly), i.e., we have

$$df(x,y) = 0 \implies f(x,y) = C. \tag{4}$$

Unfortunately not all equations can be written (with manageable ease) in the form $df = 0$, and so we need to be able to test any given ODE with form (1) to see if it is exact as it stands. We will develop a test for this in Section 7. Before doing so, we will recall partial derivatives in Section 1, look at the reverse process (integration) in Section 2, define and clarify exact differentials in Section 3, and connect them to level curves in Sections 4 and 5. This will lead to a method for solving exact ODE's which we develop in the final section.

---

[1] This makes more sense, perhaps, if we recall where this came from. Suppose we have a funcion $f(x,y)$, and a path $(x(t), y(t))$, and we want to know how the function changes along the path. According to the chain rule,

$$\begin{aligned}\frac{d}{dt}f(x(t),y(t)) &= \left.\frac{\partial f}{\partial x}\right|_{(x(t),y(t))} \cdot \frac{dx(t)}{dt} + \left.\frac{\partial f}{\partial y}\right|_{(x(t),y(t))} \cdot \frac{dy(t)}{dt} \\ &= \frac{\partial f}{\partial x}(x(t),y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t)) \cdot \frac{dy}{dt}.\end{aligned}$$

The second line is the way it might appear in a calculus textbook. This might be summarized

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \\ &= \nabla f \cdot \frac{d}{dt}(x,y),\end{aligned}$$

where we assume x and y are functions of t, and $(x,y) = (x(t),y(t))$ is some parametrized path, and its derivative with respect to t being a velocity vector. Note that if we muliply by dt, we get (2), so (2) might be more familiar if we divide it by dt.

1 Partial Derivatives

In this lecture we will assume that we have a function of two variables, namely $f(x, y)$.[2] Such a function will likely vary in both x and y, so we have **partial derivatives** to measure these variations:

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \tag{5}$$

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \tag{6}$$

Notice that in the first limit, the y-variable is fixed, i.e., *held constant*, and the same is true of the x-variable in the second limit. If we hold y fixed, then $f(x, y)$ becomes a function of x alone, and f_x measures how f changes with x. Similarly if we hold x fixed, then $f(x, y)$ becomes a function of y alone, with f_y measuring how it changes with y.

Computing these partial derivatives is just a matter of treating the other, fixed variable as a constant and using the usual differentiation rules.

Example 1 *If $f(x, y) = \sin x \cdot e^{xy^2}$, then to compute $\partial f/\partial x$ we hold y constant and take the derivative with respect to x. Similarly to find $\partial f/\partial y$ we take x to be constant and differentiate with respect to y. Thus*

$$\begin{aligned}
\frac{\partial}{\partial x}\left[f(x,y)\right] &= \sin x \cdot \frac{\partial}{\partial x}\left[e^{xy^2}\right] + e^{xy^2} \cdot \frac{\partial}{\partial x}\left[\sin x\right] \\
&= (\sin x) e^{xy^2} \frac{\partial}{\partial x}\left[xy^2\right] + e^{xy^2} \cos x \\
&= (\sin x) e^{xy^2}(y^2) + e^{xy^2} \cos x \\
&= e^{xy^2}\left(y^2 \sin x + \cos x\right), \\
\frac{\partial}{\partial y}\left[f(x,y)\right] &= \sin x \cdot e^{xy^2} \frac{\partial}{\partial y}\left[xy^2\right] \\
&= \sin x \cdot e^{xy^2} \cdot x \cdot 2y \\
&= 2xy e^{xy^2} \sin x.
\end{aligned}$$

To find $\partial f/\partial x$, we treat y no differently than any other constant, like 5 or π or 10^6. Similarly when the roles of x and y are reversed.

Since the partial differential operator $\partial/\partial x$ treats y as a constant, and $\partial/\partial y$ treats x as a constant, we have[3]

$$\frac{\partial}{\partial x}(y) = 0, \tag{7}$$

$$\frac{\partial}{\partial y}(x) = 0. \tag{8}$$

If we have a function of y only, or of x only, we likewise have[4]

$$\frac{\partial}{\partial x}\phi(y) = 0, \tag{9}$$

$$\frac{\partial}{\partial y}\psi(x) = 0. \tag{10}$$

---

[2] Much of what we do here will also work for more variables, i.e., functions $f(x, y, z)$ and so on.

[3] Note that $\partial y/\partial x$ is very different from dy/dx. This is because x and y are both taken to be *independent* variables, as in $z = f(x, y)$ when we are using partial derivatives, while we assume y is a function of x (at least locally), i.e., $y = y(x)$ with ordinary derivatives. The distinction is very important, and often a source of errors even among advanced mathematics students.

This is because $\phi(y)$ does not change with x, and $\psi(x)$ does not change with y. For example, the following may first appear to be ugly calculations, but really are quite trivial because of what the partial differential operators consider constant:

$$\frac{\partial}{\partial x}\left[(y^2+1)e^{\sin y}\sqrt{\sec^2 y+1}\right]=0,$$
$$\frac{\partial}{\partial y}\left[\frac{x^2+9x-\ln(x^2+5)}{\sqrt[3]{x^4-9x+1000}}\right]=0.$$

Example 2 *Calculate $\partial f/\partial x$ and $\partial f/\partial y$ if*

$$f(x,y)=x\sin y+x^3y^2+\cos x+\tan y.$$

$$\frac{\partial f(x,y)}{\partial x}=\sin y+3x^2y^2-\sin x,$$
$$\frac{\partial f(x,y)}{\partial y}=x\cos y+x^3\cdot 2y+\sec^2 y.$$

2 Partial Derivatives and Integration

In this section we just point out one technical point which arises when we try to find $f(x,y)$ given one of its partial derivatives. The key to this section is to remember Equations (9) and (10).

For instance, suppose we know that $f_x(x,y)=x\cos x^2y$. Since this is a statement about taking a derivative of f and getting $x\cos x^2y$, it is reasonable that we somehow take an antiderivative to recover the function, at least as far as we can. For this case, we get

$$\frac{\partial f(x,y)}{\partial x}=x\cos x^2y\implies f(x,y)=\int x\cos x^2y\,dx=\frac{1}{2y}\sin x^2y+\phi(y). \tag{11}$$

This is the most general antiderivative, in x, of $x\cos x^2y$. Rather than a simple "$+C$" as we need in single-variable calculus, the general term with zero x-partial derivative is any function of y alone. A quick check verifies that

$$\frac{\partial}{\partial x}\left[\frac{1}{2y}\sin x^2y+\phi(y)\right]=\frac{1}{2y}\left(\cos x^2y\right)\cdot 2xy+0=x\cos x^2y,$$

as desired. If we are given more data about f, we may be able to determine exactly the form of $\phi(y)$ (just as we were often able to determine the value of the "C" in single-variable calculus from added data).

Note that constants *are* also functions of y (albeit trivial ones), so the solution (11) does include the "$+C$" case, built into $\phi(y)$. Also note how y acted like a constant throughout in both calculations.

---

[4] Aside from the philosophical reasons for Equations (9) and (10) given in the paragraphs, we can also see that both follow from the chain rule and (7) and (8) respectively as follow:

$$\frac{\partial}{\partial x}\phi(y)=\phi'(y)\frac{\partial y}{\partial x}=\phi'(y)\cdot 0=0,$$
$$\frac{\partial}{\partial y}\psi(x)=\psi'(x)\frac{\partial x}{\partial y}=\psi'(x)\cdot 0=0.$$

Below are some more sample integrations which illustrate these principles.

$$\begin{aligned}
\int e^{xy}\,dx &= \frac{1}{y}e^{xy}+\phi(y),\\
\int \left(xy^2-6y+9\right)\,dy &= \frac{1}{3}xy^3-3y^2+9y+\psi(x),\\
\int \frac{1}{\sqrt{1-y^2}}\,dx &= \frac{x}{\sqrt{1-y^2}}+\phi(y),\\
\int \frac{1}{\sqrt{1-y^2}}\,dy &= \sin^{-1}y+\psi(x).
\end{aligned}$$

When dealing with definite double and triple integrals in Calculus III, the extra functions $\phi(y)$ or $\psi(x)$ were usually ignored because they disappeared in exactly the same way the $+C$ disappears with definite integrals. However, for this lecture we will need to pay closer attention to these extra functions and find their exact forms.

3 Exact Differentials

If we are given a function $f(x,y)$ of two variables, we define the **exact differential**, or **total differential** of f by

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy. \tag{12}$$

This is justified when we consider the case in which x and y are both functions of a parameter, say t. If so, then $(x,y)=(x(t),y(t))$, making $f(x,y)=f(x(t),y(t))$ ultimately a function of t, at least along the path traced by $(x,y)=(x(t),y(t))$. Then we can take the t-derivative, and get the following Calc III-type chain rule:

$$\frac{df(x,y)}{dt}=\frac{\partial f(x,y)}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f(x,y)}{\partial y}\cdot\frac{dy}{dt}. \tag{13}$$

Hence the chain rule (13) is the absolute differential (12) divided by dt. What is interesting about the exact differential (12) is that its form does not depend upon the path; we can calculate its form from that of f without reference to any parametrization.

Example 3 *Below are some simple calculations of exact differentials.*

1. *If* $f(x,y)=x^2+y^3$, *then* $df=2x\,dx+3y^2\,dy$.
2. *If* $f(x,y)=\sin x\cos y$, *then* $df=\cos x\cos y\,dx-\sin x\sin y\,dy$.

4 Level Curves and Total Differentials

Recall that many curves in the xy-plane are given as level curves of functions, and are thus written (implicitly) as $f(x,y)=C$. If we parametrize such a curve, say $x=x(t)$, $y=y(t)$, where x and y are differentiable, then we have

$$f(x(t),y(t))=C \tag{14}$$

holds for all t in the domains of $x(t)$, $y(t)$. Thus the LHS of (14) is constant in t, and will have t-derivative zero (just apply $\frac{d}{dt}$ to both sides of (14)):

$$\frac{d}{dt}\left[f(x(t),y(t))\right]=0. \tag{15}$$

Now we expand the LHS of (15) using the chain rule (13) which gives us, for all t,

$$\frac{\partial f(x(t), y(t))}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x(t), y(t))}{\partial y} \cdot \frac{dy}{dt} = 0. \tag{16}$$

Now this is true regardless of the parametrization. In fact, by multiplying by dt we see that we get

$$\frac{\partial f(x,y)}{\partial x}\, dx + \frac{\partial f(x,y)}{\partial y}\, dy = 0. \tag{17}$$

We recognize LHS of (17) as a total, i.e., exact differential of f, and so (17) becomes

$$df(x,y) = 0. \tag{18}$$

As an aside, notice that we can get from (17) an equation for dy/dx along that level curve:

$$\frac{\partial f(x,y)}{\partial y}\, dy = -\frac{\partial f(x,y)}{\partial x}\, dx \implies \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Many calculus textbooks write this using the counter-intuitive (but nonetheless correct) formula $dy/dx = -f_x/f_y$. The only reason to mention this here is that, for some cases, we can take a problem about dy/dx, work backwards algebraically and turn it into a question about total differentials.

5 Zero Total Differentials and Level Curves

A kind of converse of (18) is the following. If we are given an ODE of the form $df = 0$, then the solution will be of the form $f(x,y) = C$, which is a one-parameter family of implicit curves. To highlight this we will tag it as a separate "equation":

$$df(x,y) = 0 \implies f(x,y) = C. \tag{19}$$

6 A Brief Note on Notation and Higher-Order Partial Derivatives

We have used two different notations for partial derivatives. For instance,

$$f_x(x,y) = \frac{\partial f(x,y)}{\partial x}, \qquad f_y(x,y) = \frac{\partial f(x,y)}{\partial y}.$$

Both notations can be seen as applying a partial differential operator to f. Unfortunately, the notations are different in the orders that the variables appear, which is due to locations of the notations for the operations. For instance,

$$f_x(x,y) = (f(x,y))_x \qquad \text{and} \qquad f_y(x,y) = (f(x,y))_y.$$

Contrast this with the other notation:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[f(x,y)\right], \qquad \text{and} \qquad \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left[f(x,y)\right].$$

The usual place for confusion is with the higher-order "mixed" partial derivatives. For instance,

$$f_{xy}(x,y) = (f_x(x,y))_y = \frac{\partial}{\partial y}\left[\frac{\partial f(x,y)}{\partial x}\right] = \frac{\partial^2 f(x,y)}{\partial y \partial x}.$$

In both cases, we take the partial derivative with respect to x first, and then take the partial derivative—of the result—with respect to y.

Recall from Calculus III that if $f(x,y)$ has all continuous second-order partial derivatives f_{xx}, f_{yy}, f_{xy} and f_{yx}, then the mixed partial derivatives are the same:

Theorem 1 *If $R = \{(x,y) : x \in (a,b), y \in (c,d)\}$ is an open rectangle in the xy plane, then*

$$\left.\begin{array}{l} f_{xx} \text{ continuous in } R \\ f_{yy} \text{ continuous in } R \\ f_{xy} \text{ continuous in } R \\ f_{yx} \text{ continuous in } R \end{array}\right\} \qquad \Longrightarrow \qquad f_{xy} = f_{yx} \quad \text{in } R. \tag{20}$$

Most of the functions we deal with are continuous, and have nice, continuous partial derivatives of all orders, except where undefined or otherwise obviously misbehaving. The following very simple example shows this theorem in action. A more exhaustive collection can be found in most calculus textbooks.

Example 4 *Suppose $f(x,y) = e^{x^2} \sin y$. Then*

$$\begin{aligned} f_x(x,y) &= e^{x^2} \cdot 2x \sin y &&\Longrightarrow &&f_{xy}(x,y) = (f_x(x,y))_y = e^{x^2} \cdot 2x \cos y; \\ f_y(x,y) &= e^{x^2} \cos y &&\Longrightarrow &&f_{yx}(x,y) = (f_y(x,y))_x = e^{x^2} \cdot 2x \cos y. \end{aligned}$$

7 Testing for Exactness

Now we collect everything from before and apply it to some ODE's. First we will look at the abstract problem, and then we will do specific examples.

Suppose we are given an ODE of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{21}$$

To test if the LHS is exact, i.e., can be written df, we identify what corresponds to what. Recall that

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{22}$$

Then we must have $M(x,y) = \partial f/\partial x$, and $N(x,y) = \partial f/\partial y$:

$$\underbrace{M(x,y)}_{\partial f/\partial x}\,dx + \underbrace{N(x,y)}_{\partial f/\partial y}\,dy = 0.$$

If f has continuous second partials, i.e., M and N have continuous first partials, then

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) \tag{23}$$

becomes

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y), \tag{24}$$

or

$$M_y = N_x.$$

In fact, this is given as a theorem in Zill (page 68):

Theorem 2 *If $M(x,y)$ and $N(x,y)$ are continuous, along with their first partial derivatives, in some rectangular region $\{(x,y) : x \in (a,b), y \in (c,d)\}$, then*

$$M(x,y)\,dx + N(x,y)\,dy = df(x,y), \text{ some function } f \Longleftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{25}$$

The challenge then is to find $f(x,y)$. This is not difficult, but requires some care. It is not difficult because we know, for instance that $M(x,y) = f_x(x,y)$, and so

$$f(x,y) = \int M(x,y)\,dx,$$

except that our "constant" of integration is then some $\phi(y)$. To find $\phi(y)$, we then use the fact that

$$N(x,y) = f_y(x,y) = \frac{\partial}{\partial y} f(x,y).$$

Thus we just need to integrate M with respect to x, and reconcile the answer with the notion that its y-partial derivative will be $N(x,y)$.

Example 5 *Consider the ODE*

$$2x\sin y\,dx + (y^3 + x^2\cos y)\,dy = 0.$$

Here $M(x,y) = 2x\sin y$ *and* $N(x,y) = y^3 + x^2\cos y$. *To test for exactness we compute:*

$$\left.\begin{array}{ll} (\text{think } (f_x)_y) & \dfrac{\partial}{\partial y} M(x,y) = 2x\cos y \\ (\text{think } (f_y)_x) & \dfrac{\partial}{\partial x} N(x,y) = 2x\cos y \end{array}\right\} \Longrightarrow M_y = N_x \therefore \textit{ ODE is exact.}$$

Since the ODE is exact, we can go about finding the function $f(x,y)$ *with*

$$\begin{aligned} df &= \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy \\ &= 2x\sin y\,dx + (y^3 + x^2\cos y)\,dy = 0. \end{aligned}$$

Since $\partial f/\partial x = 2x\sin y$, *we have*

$$\begin{aligned} f(x,y) &= \int 2x\sin y\,dx \\ &= x^2\sin y + \phi(y). \end{aligned}$$

Now we use the fact that $\partial f/\partial y = N(x,y)$:

$$\begin{aligned} && \frac{\partial f}{\partial y} &= N(x,y) \\ \Longrightarrow && x^2\cos y + \phi'(y) &= y^3 + x^2\cos y \\ \Longrightarrow && \phi'(y) &= y^3 \\ \Longrightarrow && \phi(y) &= \int y^3\,dy = \frac{1}{4}y^4 \end{aligned}$$

We usually do not bother writing the "$+C_1$*" (just as we do not write the arbitrary constant in going from dv to v in integration by parts) because its effect is already present in what will be the final answer as we will see in a moment.*

At this point, we have $f(x,y) = x^2\sin y + \phi(y)$, *i.e.,*

$$f(x,y) = x^2\sin y + \frac{1}{4}y^4.$$

Recall that the ODE was found to be of the form $df = 0$, *so the final answer is* $f(x,y) = C$, *i.e.,*

$$x^2\sin y + \frac{1}{4}y^4 = C.$$

Note that the arbitrary constant of integration for $\phi'(y) \longrightarrow \phi(y)$ *can be absorbed in the parameter* C *on the RHS of the solution to the ODE.*

The general method can be quickly outlined, but of course "the devil is in the details." Actually the details are usually pretty routine, once all the parts of the method are kept straight. The method can be outlined as given below:

1. Put the equation into the form $M(x,y)\,dx + N(x,y)\,dy = 0$.
2. Check for exactness: Is $M_y = N_x$? (This is necessary for f to exist, and just states $f_{xy} = f_{yx}$.) If not, the equation is inexact and you must try something else.
3. If yes, i.e., the equation is exact, do **one** of the following:

   Option (a) Integrate M with respect to x. Since $f_x = M$, we have:

   $$f(x,y) = \int M(x,y)\,dx.$$

   Be sure to include the "constant of integration" $\phi(y)$.

   Option (b) Integrate N with respect to y. Since $f_y = N$, we have:

   $$f(x,y) = \int N(x,y)\,dy.$$

   Be sure to include the "constant of integration" $\psi(x)$.

   (The names ϕ, ψ do not matter, but the forms do. Zill uses $g(x)$ or $g(y)$, for instance.)
4. Find the final form of f by using either
   (a) $f_y = N(x,y)$ to find the form of $\phi(y)$; or
   (b) $f_x = M(x,y)$ to find the form of $\psi(x)$.
5. Once the final form of f is found, so the original ODE can be rewritten $df = 0$, the solution is therefore the one-parameter family of curves given by

   $$f(x,y) = C.$$

(At this point it would be useful to the reader to re-examine the previous example.)

Of course, the given equation may have been separable from the beginning, in which case separation methods would be preferable. In fact, all separable equations are exact, since they are of the form

$$M(x)\,dx + N(y)\,dy = 0,$$

which gives trivially that $M_y = 0 = N_x$. However, exact equations are only occasionally separable so the more general method outlined above is needed for those cases.

Homework 4-A

1. Fill out the following (some of which are contained within these notes):

   (a) $\dfrac{\partial x}{\partial x} =$

   (b) $\dfrac{\partial x}{\partial y} =$

   (c) $\dfrac{\partial y}{\partial x} =$

   (d) $\dfrac{\partial y}{\partial y} =$

   (e) $\dfrac{\partial \psi(x)}{\partial x} =$

   (f) $\dfrac{\partial \psi(x)}{\partial y} =$

   (g) $\dfrac{\partial \phi(y)}{\partial x} =$

   (h) $\dfrac{\partial \phi(y)}{\partial y} =$

2. Calculate f_x and f_y if $f(x,y) = \ln\sqrt{x^2+y^2}$.

3. Calculate f_x and f_y if $f(x,y) = x\sin(xy^2)$.

4. For $f(x,y) = x^2 \ln y + \sec x + \tan^{-1} y$, show that $f_{xy} = f_{yx}$ where defined (so you can ignore the domains, etc.).

5. Show that the following equation is exact and solve it:

$$\left(e^x \sin y + 3x^2 y\right)\, dx + \left(e^x \cos y + x^3 + 2y\right)\, dy = 0.$$