Lecture 5: Functions Homogeneous of Degree Zero and ODE's

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In this lecture we will look at solving a particular type of ODE, which can be written in the form

$$\frac{dy}{dx} = f(x,y) \tag{1}$$

where $f(x,y)$ is a particular type of function, namely one which is homogeneous of degree zero. More general types of homogeneous functions are interesting for algebraic reasons, so we will define what that means in Section 1. Next, in Section 2 we will show how an equation (1), with f homogeneous of degree zero, can be separated and therefore solved. Section 3 summarizes the actual method, and the final section shows two examples. As usual, the method, while fairly straightforward, requires some care and alertness algebraically, and alertness to the possibility of near-trivial solutions.

In the greater scheme of ODE methods, this one is not especially difficult. It really contains one clever trick to learn, and otherwise just requires the usual care.

1 Definition of Homogeneous of Degree α

A function $f(x,y)$ of two variables is called **homogeneous of degree** α if

$$f(tx,ty) = t^{\alpha} f(x,y). \tag{2}$$

One can usually spot such functions and their degrees with very little practice. Consider the following examples:

1. $f(x,y) = x^2 + 5xy + y^2$ is homogeneous of degree 2, since

$$\begin{aligned} f(tx,ty) &= (tx)^2 + 5(tx)(ty) + (ty)^2 \\ &= t^2x^2 + t^2(5xy) + t^2y^2 \\ &= t^2(x^2 + 5xy + y^2) \\ &= t^2 f(x,y). \end{aligned}$$

2. $f(x,y) = x^4y^2 + x^3y^3 + \dfrac{y^7}{x}$ is homogeneous of degree 6, since

$$\begin{aligned} f(tx,ty) &= (tx)^4(ty)^2 + (tx)^3(ty)^3 + \frac{(ty)^7}{tx} \\ &= t^6x^4y^2 + t^6x^3y^3 + \frac{t^7y^7}{tx} \\ &= t^6\left(x^4y^2 + x^3y^3 + \frac{y^7}{x}\right) \\ &= t^6 f(x,y). \end{aligned}$$

3. $f(x,y) = \dfrac{x}{x^2+y^2} + \dfrac{1}{y}\sin(x/y)$ is homogeneous of degree -1 since

$$\begin{aligned} f(tx,ty) &= \frac{tx}{(tx)^2+(ty)^2} + \frac{1}{ty}\sin\left(\frac{tx}{ty}\right) \\ &= \frac{tx}{t^2(x^2+y^2)} + t^{-1}\frac{1}{y}\sin(x/y) \\ &= t^{-1}\left(\frac{x}{x^2+y^2} + \frac{1}{y}\sin(x/y)\right) \\ &= t^{-1}f(x,y). \end{aligned}$$

Note that homogeneity is a question of total powers of x and y. A term $x^n y^m$ would be homogeneous of degree $n+m$. It is very much like treating x and y as the same variable and seeing what power of the variable we are left with. A term like x^n/y^m would be homogeneous of degree $n-m$.

We can only add and subtract terms with the same homogeneity if we would like to preserve that homogeneity. For instance $x^2y^4 + x^3y$ would be a sum of degree 6 and degree 4, and would collectively *not* be homogeneous of any degree. Similarly $\dfrac{x^4+y^2}{xy+3x^2+9y^2}$ is not homogeneous either, since the denominator is homogeneous of degree two, while the numerator is inhomogeneous. (To be convinced, try to factor a power of t out of $f(tx,ty)$ and be left with only $f(x,y)$: it is impossible!)

Some texts only require the definition (2) to hold for $t > 0$, which allows us to call $f(x,y) = \sqrt{x^2+y^2}$ homogeneous of degree 1, since

$$f(tx,ty) = \sqrt{(tx)^2+(ty)^2} = \sqrt{t^2(x^2+y^2)} = |t|\sqrt{x^2+y^2} = tf(x,y),$$

again assuming we only consider $t > 0$. This is somewhat intuitive, because the x^2+y^2 is homogeneous of degree 2, while the radical represents a one-half power. In algebra we learn that $(a^n)^m = a^{nm}$, so this kind of multiplication of degrees mimics that rule.

For the ODE technique we are interested in for this lecture we consider functions $f(x,y)$ which are **homogeneous of degree zero,** formally meaning that $f(tx,ty) = t^0 f(x,y)$, but we will assume the following for all $t \neq 0$ (even though t^0 really only makes sense if $t > 0$):

$$f(tx,ty) = f(x,y). \tag{3}$$

These are easy to spot as well. Functions such as $f(x,y) = 5x/y$, $f(x,y) = \frac{x^2+y^2}{x^2-y^2}$, and the like are such functions, for if we replace x, y with tx, ty, we do not change the function values. Another such function would be $f(x,y) = \frac{x^3y+6x^2y^2}{x^4-y^4} + \tan(x/y) + 5$.

One interesting (and later crucial) algebraic result regarding functions homogeneous of degree zero is the following:

Theorem 1 *If $f(x,y)$ is homogeneous of degree zero, then we can write $f(x,y)$ as a function of the ratio y/x, i.e.,*

$$f(x,y) = g\left(\frac{y}{x}\right). \tag{4}$$

In other words, if you know y/x (i.e., know which line through the origin (x,y) lies on), then you know $f(x,y)$. For the proof, note that

$$f(x,y) = f\left(x\cdot 1, x\cdot\frac{y}{x}\right) = f\left(1, \frac{y}{x}\right).$$

Note that the part of t in the definition of homogeneity is played by x. Also note that the above is indeed only dependent upon y/x as claimed. Indeed, if we define a function g by

$$g(s) = f(1,s),$$

we have

$$g\left(\frac{y}{x}\right) = f\left(1, \frac{y}{x}\right) = f(x, y).$$

That completes the proof.[1]

A proof like the above can require some pondering before it loses its sleight-of-hand appearance and finally seems believable, but it is correct and an examination will find no errors.

2 Theory and Separation

In this short section we show how the equation

$$\frac{dy}{dx} = f(x, y), \qquad f \text{ homogeneous of degree zero,} \tag{5}$$

can be separated. The final method will be contained in this derivation, and summarized at the start of Section 3. It is basically a substitution-type argument. We give its derivation here.

So we go back to our ODE (5), which with the help of Theorem 1 we can now write as follows:

$$\frac{dy}{dx} = g(y/x). \tag{6}$$

The next, and key step is to make a substitution:

$$u = \frac{y}{x}. \tag{7}$$

Actually the more useful form will be the following:

$$y = ux. \tag{8}$$

Now we differentiate (8) with respect to x, i.e., apply $\frac{d}{dx}$ to both sides. Using the product rule we get $\frac{dy}{dx} = u\frac{dx}{dx} + x\frac{du}{dx}$, i.e.,

$$\frac{dy}{dx} = u + x\frac{du}{dx}. \tag{9}$$

With this our ODE, as in (6) becomes

$$u + x\frac{du}{dx} = g(u). \tag{10}$$

This is easily separated, the first steps being

$$(10) \implies \qquad x\frac{du}{dx} = g(u) - u, \tag{11}$$

$$\implies \qquad \frac{du}{dx} = \frac{g(u) - u}{x}. \tag{12}$$

[1] A function in x and y, homogeneous of degree zero, and therefore a function of y/x, has some interesting geometric properties.

1. If we write the function instead in polar coordinates, say $f(x, y) = \phi(r, \theta)$, then it is constant along lines $\theta = \theta_0$, i.e., $y/x = \tan\theta_0$, and so $f(x, y) = \phi(r, \theta) = \Phi(\theta)$, and not depending upon the radial coordinate. So $f(x, y)$ is just a function of the polar angle θ.
2. Therefore, unless that function is constant (which is one example of a degree-zero homogeneous function), it is necessarily discontinuous at the origin $(x, y) = (0, 0)$, i.e., $r = 0$.
3. If x and y are given the same dimensions, such as feet, then $f(x, y)$ will be dimensionless (feet cancel). This is worth noting as we work through examples later.

At this point we have to be careful to include those cases of constant u, i.e., $u = k$ which cause the RHS of (12) to be zero (and recall that $u = k$ would make LHS zero as well), and are thus more trivial solutions to the new ODE. Putting those aside for the moment, we instead separate and put in our integral signs to get a solution in u and x:

$$\int \frac{du}{g(u) - u} = \int \frac{dx}{x} \tag{13}$$

$$\implies \qquad G(u) = \ln|x| + C, \tag{14}$$

where $G(u)$ is some antiderivative of $1/(g(u) - u)$ in u, i.e., $G'(u) = 1/(g(u) - u)$. The last thing to do is replace u using $u = y/x$, giving us

$$G\left(\frac{y}{x}\right) = \ln|x| + C, \tag{15}$$

as well as the $u = k$ solutions to (12), i.e., $y/x = k$, i.e.,

$$y = kx \tag{16}$$

solutions. Of course there may be some desirable algebra to include in order to simplify the presentation of the general solution, such as applying the exponential function exp to both sides of (15) to rid ourselves of the natural log in the RHS.[2]

3 Method and Examples

A method is embedded in the previous section. Assuming we have already written the equation into the form (1), i.e.,

$$\frac{dy}{dx} = f(x, y)$$

we then do the following:

1. Verify that $f(x, y)$ is homogeneous of degree zero. (If not, try to find another method.) In other words, check
$$f(tx, ty) = f(x, y).$$
2. Substitute $y = ux$ in LHS and RHS of the ODE.
   (a) LHS becomes $u + x\dfrac{du}{dx}$.
   (b) RHS will become a function of u alone. (There are two techniques for accomplishing this.)
3. Solve algebraically for $\dfrac{du}{dx}$, noting the possible "constant u" solutions.
4. Separate and solve the new form of the ODE.
5. Replace $u = y/x$ in all solutions. DONE (except for checking solutions make sense in original ODE, which is rarely not the case with these particular ODE's).

[2]One really should check any solutions in the original ODE (5) to be sure, through all the divisions etc., that such y's do not cause RHS of the original ODE to be undefined, for instance. There are also cases where the constant u solutions, i.e., $y = kx$ solutions reappear when we simplify (15), and possibly cases where these give dy/dx to be undefined in the original ODE. As a general rule the answer has to be consistent with the original ODE at some points. Indeed, when we perform algebraic steps in between calculus steps, for instance, there can be the danger that a problem in the original ODE is lost in those calculations, or a more trivial solution is lost in the same calculations.

Example 1 *Solve the (non-separable, nonexact, nonlinear) ODE*

$$\frac{dy}{dx} = \frac{x+y}{x-y}. \tag{17}$$

First we can see $f(x) = (x+y)/(x-y)$ *is homogeneous of degree zero, though we could check:*

$$f(tx,ty) = \frac{tx+ty}{tx-ty} = \frac{t(x+y)}{t(x-y)} = \frac{x+y}{x-y} = f(x,y).$$

That being verified (or just noted), we now substitute $y = ux \implies \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$.

$$\begin{aligned} \frac{d(ux)}{dx} &= \frac{x+ux}{x-ux} \\ \implies u + x\frac{du}{dx} &= \frac{x(1+u)}{x(1-u)} \\ \implies u + x\frac{du}{dx} &= \frac{1+u}{1-u}. \end{aligned}$$

Solving for $\frac{du}{dx}$ *(and then combining RHS into one fraction) gives*

$$x\frac{du}{dx} = \frac{1+u}{1-u} - u = \frac{1+u-u(1-u)}{1-u} = \frac{u^2+1}{1-u}.$$

Thus we must solve

$$x\frac{du}{dx} = \frac{u^2+1}{1-u}. \tag{18}$$

After separation we get

$$\begin{aligned} && \frac{1-u}{u^2+1}\,du &= \frac{dx}{x} \\ \implies && \int \left(\frac{1}{u^2+1} - \frac{u}{u^2+1}\right) du &= \int \frac{dx}{x} \\ \implies && \tan^{-1} u - \frac{1}{2}\ln(u^2+1) &= \ln|x| + C_1. \end{aligned}$$

Substituting $u = y/x$ *then gives us*

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(\left(\frac{y}{x}\right)^2 + 1\right) = \ln|x| + C_1. \tag{19}$$

One algebraic step which could now be taken would be to combine the fraction inside the log and rewrite the log on the RHS as $\ln(x^2)^{1/2} = \frac{1}{2}\ln x^2$*:*

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\frac{y^2+x^2}{x^2} = \frac{1}{2}\ln x^2 + C_1. \tag{20}$$

Expanding the log on our current LHS gives

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(y^2+x^2) + \frac{1}{2}\ln x^2 = \frac{1}{2}\ln x^2 + C_1. \tag{21}$$

We see that the $\ln x^2$ *terms cancel, and we get*

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(y^2+x^2) = C. \tag{22}$$

(There were no constant u cases in this example.)

Before we finish this example, we note an alternative method for writing the RHS of the ODE in terms of u:

$$\frac{x+y}{x-y}\cdot\frac{\frac{1}{y}}{\frac{1}{y}}=\frac{\frac{x}{y}+1}{\frac{x}{y}-1}=\frac{u+1}{u-1}.$$

Technically the above example was solved at (19), but a little algebraic ingenuity helped simplify the solution. However the solution, as finally presented in (22) is still given by a family of *level curves* of a function, i.e., $F(x,y)=C$ where $F(x,y)=\tan^{-1}(y/x)-\frac{1}{2}\ln(y^2+x^2)$. As we've seen before, this is very typical with ODE's; we are lucky when we can solve for $y=y(x)$!

Example 2 *Solve the ODE*

$$\frac{dy}{dx}=\frac{2y^2-22xy}{xy+3x^2}. \tag{23}$$

First we check that $f(x,y)=(2y^2-22xy)/(xy+3x^2)$ is homogeneous of degree zero. It is, as the calculation below confirms (again note it is reasonable to check visually; see Item 3 in Footnote 1).

$$f(tx,ty)=\frac{2t^2y^2-22txty}{txty+3t^2x^2}=\frac{t^2(2y^2-22xy)}{t^2(xy+3x^2)}=\frac{2y^2-22xy}{xy+3x^2}=f(x,y).$$

Next we substitute $y=xu$, and so $y=ux \implies \dfrac{dy}{dx}=u+x\dfrac{du}{dx}$.

$$\begin{aligned}
u+x\frac{du}{dx}&=\frac{2(ux)^2-22x(ux)}{x(ux)+3x^2}\\
\implies u+x\frac{du}{dx}&=\frac{x^2(2u^2-22u)}{x^2(u+3)}\\
\implies u+x\frac{du}{dx}&=\frac{2u^2-22u}{u+3}\\
\implies \quad x\frac{du}{dx}&=\frac{2u^2-22u}{u+3}-u\\
\implies \quad x\frac{du}{dx}&=\frac{2u^2-22u-u^2-3u}{u+3}\\
\implies \quad x\frac{du}{dx}&=\frac{u^2-25u}{u+3}\\
\implies \quad \frac{du}{dx}&=\frac{1}{x}\cdot\frac{u^2-25u}{u+3}.
\end{aligned}$$

(Note the alternative method for computing the RHS of (23) as a function of u, here namely multiplying numerator and denominator by $1/y^2$.) From here we see that $u(u-25)=0$ gives zero on the right, so $u=0,25$ are both solutions to this new ODE. Now we look for others by separation:

$$\begin{aligned}
\frac{u+3}{u^2-25u}\,du&=\frac{1}{x}\,dx\\
\implies \int\frac{u+3}{u^2-25u}\,du&=\int\frac{1}{x}\,dx.
\end{aligned}$$

The integral on the left requires partial fractions:

$$\begin{aligned}
\frac{u+3}{u(u-25)}&=\frac{A}{u}+\frac{B}{u-25}\\
u+3&=A(u-25)+Bu\\
\underline{u=0}: \qquad 3&=-25A \implies \boxed{A=-3/25}.\\
\underline{u=25}: \qquad 28&=25B \implies \boxed{B=28/25}.
\end{aligned}$$

This all gives us

$$\int \left(\frac{-3/25}{u} + \frac{28/25}{u-25} \right) du = \int \frac{dx}{x}.$$

The solution in u, x *is then*

$$-\frac{3}{25} \ln |u| + \frac{28}{25} \ln |u-25| = \ln |x| + C_1.$$

One thing that we can do here is combine all the logarithms:

$$\begin{aligned} \frac{1}{25} \ln \left| \frac{(u-25)^{28}}{u^3} \right| &= \ln |x| + C_1 \\ \implies \quad \ln \left| \frac{(u-25)^{28}}{u^3} \right| &= 25 \ln |x| + 25C_1 \\ \implies \quad \ln \left| \frac{(u-25)^{28}}{x^{25}u^3} \right| &= C_2 \end{aligned}$$

Taking exponentials (and combining $+/-$ *cases) we get*

$$\frac{(u-25)^{28}}{x^{25}u^3} = C.$$

Replacing $u = y/x$ *gives*

$$C = \frac{\left(\frac{y}{x} - 25\right)^{28}}{x^{25}\left(\frac{y}{x}\right)^3} = \frac{\left(\frac{1}{x}\right)^{28}(y-25x)^{28}}{x^{22}y^3} = \frac{(y-25x)^{28}}{x^{50}y^3}.$$

Combining this with the constant u*-solutions* $u = 0, 25$ *gives us*

$$(y-25x)^{28} = Cx^{50}y^3, \tag{24}$$

$$y = 0, \qquad y = 25x. \tag{25}$$

If we check the original equation (23) *we see that* $y = 0$ *is a solution, and nothing terrible goes wrong if we "plug in"* $y = 25x$ *so we can be sure that is also a solution (without needing to go through all the calculations to prove it). In fact, the* $y = 25x$ *solution is the* $C = 0$ *case in* (24)*, but we can not see the* $y = 0$ *solution from that form (so Zill would call it a singular solution). The moral is that we need to find these linear solutions* $y = kx$ *from the equation for* du/dx *because they may not (all) appear in the final formula from separation of variables calculations.*

Homework 5-A

Show the RHS is homogeneous of degree zero and solve the following ODE:

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}.$$