Lecture 6: Linear ODE's with Constant Coefficients I

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1 LHODEs, and Linear Operators Revisited

In this and the next few lectures we will be interested in *linear, homogeneous* ordinary differential equations, or LHODE's, with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{1}$$

(Recall $y^{(k)}$ is the kth derivative of y.) In this context, the underlined words above mean the following:

1. **linear:** The LHS can be written as $L[y]$, where L is a linear differential operator.
2. **homogeneous:** The RHS is equal to zero (assuming LHS $= L[y]$).

We defined linear differential operators in Lecture 3. Here we briefly recall that discussion, in the context of constant coefficients (meaning the a_i's do not vary with x, y, etc.). We can define the LHS of (1) by

$$\begin{aligned} L[y] &= a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y \\ &= \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) y. \end{aligned}$$

Recall that $D = \frac{d}{dx}$, so $D^k = \frac{d^k}{dx^k}$ is the operator which takes the kth derivative with respect to x. That L is linear means that

$$L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2] \tag{2}$$

for any functions y_1, y_2 and constants (a.k.a. *scalars*) $\alpha, \beta \in \mathbb{R}$. Recall that this was a summary of two facts about linear operators, namely that they preserve addition and scalar multiplication:

$$L[y_1 + y_2] = L[y_1] + L[y_2], \tag{3}$$

$$L[\alpha y] = \alpha L[y]. \tag{4}$$

This linearity will be important in this lecture, and crucial in later lectures when we also consider nonhomogeneous equations.

2 Solution of $(D+k)y=0$

Here we will derive the solution of equations of the form $(D+k)y=0$. Actually this is separable:

$$\begin{aligned}
& & (D+k)y &= 0 \\
&\implies & \left(\frac{d}{dx}+k\right)y &= 0 \\
&\implies & \frac{dy}{dx}+ky &= 0 \\
&\implies & \frac{dy}{dx} &= -ky \qquad \text{(note } y=0 \text{ is a trivial solution)} \\
&\implies & \frac{dy}{y} &= -k\,dx \\
&\implies & \int\frac{dy}{y} &= \int(-k)\,dx \\
&\implies & \ln|y| &= -kx+C_1 \\
&\implies & |y| &= e^{-kx}e^{C1} \\
&\implies & y &= Ce^{-kx}, \quad C\in\mathbb{R} \qquad \text{(since } C=0 \text{ works too)}.
\end{aligned}$$

Note that we get a one-parameter solution of curves for the first-order ODE $\frac{dy}{dx}+ky=0$. This is really quite general, for even if $k=0$ we are solving $dy/dx=0$, which has solutions $y=C$, or $y=Ce^{0x}$. Thus we showed that, for any constant k, we have

$$(D+k)y=0 \qquad \iff \qquad y=Ce^{-kx}. \tag{5}$$

Example 1 *To see the pattern, consider the following ODE's and their solutions:*

$$\begin{array}{lrcl}
1. & (D-1)y=0 & \iff & y=Ce^{x}; \\
2. & (D+1)y=0 & \iff & y=Ce^{-x}; \\
3. & (D-2)y=0 & \iff & y=Ce^{2x}; \\
4. & (D+2)y=0 & \iff & y=Ce^{-2x}; \\
5. & \left(D+2-\sqrt{3}\right)=0 & \iff & y=e^{\left(-2+\sqrt{3}\right)x}; \\
6. & Dy=0 & \iff & y=C.
\end{array}$$

It is important to be able to read these backwards as well as forwards, for later we will want to find the differential operators (hopefully of form $(D+k)$) which will *annihilate* functions. For instance, given a function $y=Ce^{5x}$, it is useful to know that

$$(D-5)Ce^{5x}=0.$$

3 Algebra of Operators $(D+k)$, $(D+l)$

One very nice fact about differential operators with constant coefficients is that they commute, meaning

$$\left[(D+k)(D+l)\right]y=\left[(D+l)(D+k)\right]y. \tag{6}$$

The proof is as follows:

$$\begin{aligned}
\text{LHS} &= [(D+k)(D+l)]\,y \\
&= (D+k)(Dy+ly) \\
&= (D+k)(y'+ly) \\
&= D(y'+ly)+k(y'+ly) \\
&= Dy'+Dly+ky'+kly \\
&= y''+ly'+ky'+kly \\
&= y''+(l+k)y'+(kl)y. \\
\text{RHS} &= [(D+l)(D+k)]\,y \\
&= (D+l)(Dy+ky) \\
&= (D+l)(y'+ky) \\
&= D(y'+ky)+l(y'+ky) \\
&= Dy'+Dky+ly'+lky \\
&= y''+ky'+ly'+lky \\
&= y''+(k+l)y'+(lk)y.
\end{aligned}$$

So we see LHS = RHS. Note that in both cases we also get

$$[(D+k)(D+l)]\,y = \left[D^2+(k+l)D+(lk)\right]y. \tag{7}$$

Thus we can manipulate these operators much the same as polynomials, so long as they are left of the function y.

The same principles used to prove (6) can be used to show that any two linear operators with constant coefficients will commute:[1]

$$\begin{aligned}
&\left[\left(a_nD^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_0\right)\left(b_mD^m+b_{m-1}D^{m-1}+\cdots+b_1D+b_0\right)\right]y \\
&\qquad = \left[\left(b_mD^m+b_{m-1}D^{m-1}+\cdots+b_1D+b_0\right)\left(a_nD^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_0\right)\right]y.
\end{aligned}$$

4 Principle of Superposition

In this section we point out a part of the general theory of linear homogeneous ODE's, all of which be written in the form $L[y]=0$ where L is a linear differential operator.

1. If a linear differential operator annihilates a function y_1, then it also annihilates ay_1 for any constant $a\in\mathbb{R}$.

   To see this, just note that $L[ay_1]=aL[y_1]=0$ if L annihilates y_1, i.e., if $L[y_1]=0$.

2. If a linear differential operator annihilates functions y_1 and y_2, then it also annihilates y_1+y_2.

   To see this, note that if $L[y_1]=0$ and $L[y_2]=0$, then $L[y_1+y_2]=L[y_1]+L[y_2]=0+0=0$.

[1]This is not true if we have linear operators with nonconstant coefficients. For example,

$$\begin{aligned}
(D+x)(D+\sin x)y &= (D+x)(y'+\sin x\cdot y) \\
&= D(y'+y\sin x)+x(y'+y\sin x) \\
&= y''+y\cos x+y'\sin x+xy'+xy\sin x, \qquad \text{while} \\
(D+\sin x)(D+x)y &= (D+\sin x)(y'+xy) \\
&= D(y'+xy)+\sin x\cdot(y'+xy) \\
&= y''+y+xy'+y'\sin x+xy\sin x.
\end{aligned}$$

We see that the first arrangement yields a $y\cos x$ where the second just has y.

This can be summarized usefully as follows:

If L annihilates y_1 and y_2, then L also annihilates any linear combination $ay_1 + by_2$:

$$L[ay_1 + by_2] = aL[y_1] + bL[y_2] = a \cdot 0 + b \cdot 0 = 0.$$

This is called the *principle of superposition.* It only requires a linear operator (not necessarily with constant coefficients) and generalizes to functions $y_1, \cdots, y_n$. In the language of equations, it states:

If $y_1, y_2, \cdots, y_n$ are solutions of a linear homogeneous ODE (think $L[y_1] = 0$, $L[y_2] = 0$, etc.), then so is any function written

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n. \tag{8}$$

Thus linear combinations of solutions are also solutions. Note that the solution $y = 0$ always satisfies $L[y] = 0$, and is usually referred to as *the trivial solution* to the linear homogeneous ODE $L[y] = 0$.[2]

It should be clear that, if L is linear, then applying L to both sides of (8) gives

$$\begin{aligned} L[y] &= L\left[c_1 y_1 + c_2 y_2 + \cdots + c_n y_n\right] \\ &= c_1 L[y_1] + c_2 L[y_2] + \cdots + c_n L[y_n] \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 = 0. \end{aligned}$$

5 Annihilating Sums of Functions

Perhaps this is best introduced by an example.

Example 2 *Find a nontrivial*[3] *linear differential operator which annihilates the function $y = e^{2x} + e^{-5x}$.*

Solution: Here we recognize that $(D-2)$ will annihilate the e^{2x} term, while $(D+5)$ would annihilate the e^{-5x} term. Now the claim is that

$$L = (D-2)(D+5)$$

will annihilate the sum of these. Here we use the commutative properties of these operators $(D+k)$. One way to see this is to digest the following three facts simultaneously:

1. $(D-2)(D+5)e^{2x} = (D+5)(D-2)e^{2x} = (D+5)(0) = 0$. *Here we used the commutativity.*
2. $(D-2)(D+5)e^{-5x} = (D-2)(0) = 0$.
3. $[(D-2)(D+5)](e^{2x} + e^{-5x}) = [(D-2)(D+5)]e^{2x} + [(D-2)(D+5)]e^{-5x} = 0 + 0 = 0$. *Here we used the linearity and the earlier two steps.*

[2] To see that the function $y = 0$ always satisfies linear homogeneous ODE's, note that

$$L[0] = L[0 \cdot 0] = 0 \cdot L[0] = 0.$$

This is an old linear algebra trick. In fact most results dealing with linear differential operators in the general sense (not restricted to constant coefficients) have their analogs in general linear algebra, since they are linear operators in the linear algebraic sense as well.

[3] The *trivial* linear differential operator is $L[y] = 0y$, which eats any function and spits out zero. It is a differential operator in the sense that it can be written as an operator with constant coefficients but with all the coefficients being zero.

If we are not convinced, we can check this more directly:

$$\begin{aligned}(D-2)(D+5)(e^{2x}+e^{-5x}) &= (D^2+3D-10)(e^{2x}+e^{-5x}) \\ &= D^2(e^{2x}+e^{-5x})+3D(e^{2x}+e^{-5x})-10(e^{2x}+e^{-5x}) \\ &= (4e^{2x}+25e^{-5x})+3(2e^{2x}-5e^{-5x})-10(e^{2x}+e^{-5x}) \\ &= e^{2x}(4+6-10)+e^{-5x}(25-15-10) \\ &= e^{2x}(0)+e^{-5x}(0)=0, \qquad \textit{q.e.d.}\end{aligned}$$

If even the above argument does not entirely convince, one could apply $(D+5)$ to the function $e^{2x}+e^{-5x}$, and then apply $(D-2)$ to the result. Here we did not reach that far back, since we showed earlier that we could "multiply out" the operators first and then apply them to the functions.

Example 3 *Find a nontrivial linear differential operator which annihilates the function* $y = 3e^{7x} - 5e^{-2x} + 27$.

<u>*Solution*</u>*: We need $(D-7)$ for the first term, $(D+2)$ for the second, and D for the third, constant term. Since these commute, we can write our operator as*

$$L = D(D-7)(D+2).$$

Looking ahead to the next lectures, it is interesting to note here that we not only get an annihilating operator, but we also get an ODE which the function satisfies in the process. We found that

$$D(D-7)(D+2)(3e^{7x}-5e^{-2x}+27)=0,$$

i.e.,

$$(D^3-5D^2-14D)(3e^{7x}-5e^{-2x}+27)=0.$$

This means the given function is a solution of

$$y'''-5y''-14y'=0. \tag{9}$$

In fact, since the constant multipliers $3, 5, 27$ do not change the facts that the functions e^{7x}, e^{-2x} and 1 (the function $y = 1$) are all annihilated by L, we actually have that any

$$y = ae^{7x}+be^{-2x}+c \tag{10}$$

satisfies the ODE. With observations we will make later, we will be able to see that, in fact, (10) is *the* general solution to the ODE (9). The solution to (9) is thus a three-parameter family of curves.

<u>Homework 6-A</u>

1. Find a nontrivial differential operator (as in the examples) which annihilates the given function:
   (a) $6e^{7x}-9e^{2x}+5e^{-x}$.
   (b) $10-e^x$
   (c) $e^{3x}-7e^{\left(6-\sqrt{17}\right)x}$
   (d) 5
   (e) $5+x$ (Think about the one above it, and some simple calculus.)
   (f) $5+x+x^2$
2. Show that (D^2+k^2) annihilates $\sin kx$ and $\cos kx$.
3. Find an annihilator for $5-e^{6x}+2e^{-25x}+2\cos 3x$.

4. Consider the ODE

$$y'' - 4y' - 12y = 0.$$

(a) Write the LHS using a polynomial of D, times y, as in one side of (7).

(b) Factor the operator on the LHS.

(c) Find two solutions, i.e., two distinct functions annihilated by the operator on the LHS.

(d) Speculate about the form of the general solution of this ODE.