Lecture 7: LHODEs II

February 5, 2007

1 Characteristic Equation

We saw before that we could take an LHODE (with constant coefficients) such as

$$y''' - 2y'' - 15y' = 0, \tag{1}$$

rewrite it using differential operators,

$$(D^3 - 2D^2 - 15D)y = 0, \tag{2}$$

factor the operator on the left,

$$D(D-5)(D+3)y = 0, \tag{3}$$

then solve separately

$$\begin{aligned} Dy_1 &= 0 &\iff&& y_1 &= A \\ (D-5)y_2 &= 0 &\iff&& y_2 &= Be^{5x} \\ (D+3)y_3 &= 0 &\iff&& y_3 &= Ce^{-3x}, \end{aligned}$$

and finally combine all of these to get

$$D(D-5)(D+3)y = 0 \qquad \iff \qquad y = A + Be^{5x} + Ce^{-3x}. \tag{4}$$

Associated with the operator and its factoring is what is called the **characteristic equation**, which for this case is

$$m^3 - 2m^2 - 15m = 0, \tag{5}$$

with the LHS of (5) called the **characteristic polynomial** of (1). Just as we factored the operator in (3), here we solve the characteristic equation by factoring:

$$m(m-5)(m+3) = 0 \qquad \iff \qquad m = 0, 5, -3. \tag{6}$$

Each function y_1, y_2, y_3 from before corresponds to a solution of the characteristic equation. Recall from before that linear combinations of solutions of an LHODE are also solutions, so

$$y = A + Be^{5x} + Ce^{-3x}$$

will be a solution, and as we will see later, is *the* solution to the original LHODE. (Note $A = Ae^{0x}$.)

Example 1 *Use the characteristic equation to solve $y'' + 3y' - 40y = 0$.*
<u>*Solution*</u>*: Here the characteristic equation is $m^2 + 3m - 40 = 0$, which gives $(m-5)(m+8) = 0$, which has solutions $m = 5, -8$, and so the solution to the LHODE is*

$$y = Ae^{5x} + Be^{-8x}.$$

We used the following fact, which is really just a reorganization of what we found in the previous lecture:

Theorem 1 *If $m = a$ is a solution to the characteristic equation of an LHODE, then $y = Ce^{ax}$ is a solution to the original LHODE.*

This is because $m = a$ being a solution to the characteristic equation is the same as $(m - a)$ being a factor of the characteristic polynomial, which is the same as $(D - a)$ being a factor of the operator acting on y in the original LHODE. Last time we showed that this implies e^{ax} is a solution (since it is annihilated by the factor $(D - a)$ in the operator), and the linear algebra gave us the Ce^{ax} is then also a solution.

Example 2 *Use the characteristic equation to solve $y'' - 3y' + y = 0$.*

Solution: The characteristic equation here is $m^2 - 3m + 1 = 0$, which requires either completing the square (which we shall skip) or the quadratic equation:

$$m = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

There are two solutions for m, each with its own set of solutions to the ODE, and putting them together gives us

$$y = A \exp\left[\left(\frac{3+\sqrt{5}}{2}\right)x\right] + B \exp\left[\left(\frac{3-\sqrt{5}}{2}\right)x\right].$$

This latest example illustrates one convenience of the characteristic equation, since actually factoring the operator in the ODE would not be convenient, though it could be done:

$$\begin{aligned} & y'' - 3y' + y = 0 \\ \iff \quad & (D^2 - 3D + 1)y = 0 \\ \iff \quad & \left(D - \frac{3+\sqrt{5}}{2}\right)\left(D - \frac{3-\sqrt{5}}{2}\right)y = 0. \end{aligned}$$

2 Linear Independence

Linear independence is another very useful linear algebraic concept.[1] Perhaps the simplest example to look at is the set of three coordinate unit vectors used in Physics and Calculus. This is the set $\left\{\vec{i}, \vec{j}, \vec{k}\right\}$ of vectors with unit length pointing in the positive x, y and z directions, respectively. All the usual Physics-style three-dimensional vectors (members of $\mathbb{R}^3$ in linear-algebra speak) can be written as linear combinations of these three. So for example if a vector is given by $\langle 2, -5, 7\rangle$, it can also be written

$$\langle 2, -5, 7\rangle = 2\vec{i} - 5\vec{j} + 7\vec{k}.$$

Of course $\vec{i} = 1\vec{i} + 0\vec{j} + 0\vec{k}$, while $\vec{j} = 0\vec{i} + 1\vec{j} + 0\vec{k}$ and $\vec{k} = 0\vec{i} + 0\vec{j} + 1\vec{k}$, and it is impossible to write $\vec{i}$ as a linear combination of $\vec{j}$ and $\vec{k}$, for instance. That is because $\vec{i}$ is somehow a direction "independent" of $\vec{j}$ and $\vec{k}$. Indeed, if we were allowed only linear combinations of $\vec{j}$ and $\vec{k}$, we would be "stuck" with vectors parallel to the yz-plane (also known as the **span** of $\left\{\vec{j}, \vec{k}\right\}$ in linear algebra). Yes, including the $\vec{i}$ direction lets us break out of the yz-plane and have vectors with an x-component. In fact, all three vectors $\vec{i}, \vec{j}, \vec{k}$ are independent of each other.

In linear algebra, "independence" refers to sets; we call any set of "vectors," say $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ **linearly independent** if none of the $\vec{v}_k$'s can be written as a linear combination of the other vectors $\vec{v}_1 \cdots, \vec{v}_{k-1}, \vec{v}_{k+1}, \cdots, \vec{v}_n$. Otherwise, we call the set **linearly dependent** (or just **dependent**).

[1] Linear independence makes the most sense when introduced while studying vector spaces, but we will not give a rigorous—or any other—definition of vector spaces here.

A very closely related concept is **dimension**. We intuitively know the space $\mathbb{R}^3$ (i.e., the xyz space) to be three-dimensional, but why? Because it takes three real variables to describe all possible positions (or position vectors, if we'd like) in that space. There is redundancy if we introduce a fourth direction within $\mathbb{R}^3$, because it would have to be a linear combination of $\vec{i}, \vec{j}$ and $\vec{k}$. So if $\{\vec{v}_1 \cdots, \vec{v}_n\}$ is independent, then the set of all linear combinations of the $\vec{v}_i$'s is an n-dimensional space, whereas the set of linear combinations of the $\vec{v}_i$'s (again, known in linear algebra as the span of $\{\vec{v}_1, \cdots, \vec{v}_n\}$) would necessarily be lower-dimensional if that set is dependent.

The description of independence given above turns out not to be the simplest to work with in linear algebra. For instance, we would have to check that each $\vec{v}_k$ could not be written as a linear combination of the others, and if we do have independence that means checking n possibilities. The definition given in linear algebra is less obvious, but equivalent and ultimately easier to work with. It is the following:

Definition 2.1 *We call a set of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$* **linearly independent** *if and only if the only solution of*

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = 0 \tag{7}$$

is the **trivial solution** *$\alpha_1, \alpha_2, \cdots, \alpha_n = 0$. If a* **nontrivial solution** *of* (7) *exists, i.e., one in which not all the α_i's are zero, then we call the set* **dependent**. *(Here $\alpha_1, \cdots, \alpha_n$ are taken from $\mathbb{R}$; they are scalars.)*

A standard linear algebra homework assignment is to show that, if (7) holds with one of the α_k being nonzero, then that corresponding $\vec{v}_k$ can be written in terms of the other vectors, and hence the definition justifies our earlier intuition of dependence. This is just because if $\alpha_k \neq 0$, we can solve (7) for $\vec{v}_k$ (with (7) implying the first line and $a_k \neq 0$ giving the second):

$$\begin{aligned} & \alpha_1 \vec{v}_1 + \cdots + \alpha_{k-1} \vec{v}_{k-1} + \alpha_{k+1} \vec{v}_{k+1} + \cdots + \alpha_n \vec{v}_n = -\alpha_k \vec{v}_k \\ \implies \quad & -\left(\frac{\alpha_1}{\alpha_k}\right) \vec{v}_1 - \cdots - \left(\frac{\alpha_{k-1}}{\alpha_{k-1}}\right) \vec{v}_{k-1} - \left(\frac{\alpha_{k+1}}{\alpha_{k+1}}\right) \vec{v}_{k+1} - \cdots - \left(\frac{\alpha_n}{\alpha_k}\right) \vec{v}_n = \vec{v}_k. \end{aligned}$$

The reason we care about linear independence is that we need to know, when we look for the general solution of an LHODE, what the dimension of the solution family should be (that part will be easy), and whether we have enough independent functions to properly span that set (that part will be harder to prove, but usually easy to see). A well known trick is to use a device called the **Wronskian**.[2] This somewhat brute-force method for detecting independence can be easily derived from linear algebraic principles, so we will do that here for the interested reader. Of course Theorem 2 the end of the section is the crucial point.

Suppose we have n functions $f_1, f_2, \cdots, f_n$ and we would like to know if they are linearly independent. In other words, we would like to see if the equation

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x) = 0 \tag{8}$$

has nontrivial solutions.[3]

The clever idea used in deriving the Wronskian is that if the LHS is identically zero, and the functions themselves have first, second, and up to $(n-1)$ derivatives, we can differentiate (8) repeatedly to fill out a system of n equations with n unknowns:

$$\begin{array}{cccccccccc} & \alpha_1 f_1(x) & + & \alpha_2 f_2(x) & + & \cdots & + & \alpha_n f_n(x) & = & 0 \\ \implies & \alpha_1 f_1'(x) & + & \alpha_2 f_2'(x) & + & \cdots & + & \alpha_n f_n'(x) & = & 0 \\ \implies & \alpha_1 f_1''(x) & + & \alpha_2 f_2''(x) & + & \cdots & + & \alpha_n f_n''(x) & = & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \implies & \alpha_1 f_1^{(n-1)}(x) & + & \alpha_2 f_2^{(n-1)}(x) & + & \cdots & + & \alpha_n f_n^{(n-1)}(x) & = & 0 \end{array} \tag{9}$$

[2] Józef Wroński, Polish mathematician, 1778–1853.

[3] It is important to note that we are looking for numbers $\alpha_1, \cdots, \alpha_n$ which make the LHS of (8) *identically zero*, in the same way that $1 \cdot \sin^2 x + 1 \cdot \cos^2 x - 1 \cdot 1 = 0$, that is, one set of α_i's, not all zero, which makes the LHS of (8) zero no matter what x.

A fact from linear algebra is that the system above will have nontrivial solutions only if the following (square, as required by linear algebra) determinant is identically zero (zero for all x in our domain):[4]

$$W\left(f_1(x), f_2(x), \cdots, f_n(x)\right) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}. \tag{10}$$

The contrapositive to [*dependence* $\Longrightarrow$ $W = 0$] is [$W \neq 0 \Longrightarrow$ *independence*]:

Theorem 2 *If $f_1, f_2, \cdots, f_n$ are $(n-1)$-times differentiable, and $W\left(f_1(x), f_2(x), \cdots, f_n(x)\right)$ is not identically zero, then $\{f_1, f_2, \cdots, f_n\}$ is a linearly independent set of functions.*

For instance, in Example 1 we saw that $y = Ae^{5x} + Be^{-8x}$ satisfies the ODE $y'' + 3y' - 40y = 0$. To see that e^{5x}, e^{-8x} are linearly independent, we compute the Wronskian[5]

$$\begin{aligned} W\left(e^{5x}, e^{-8x}\right) &= \begin{vmatrix} e^{5x} & e^{-8x} \\ \frac{d}{dx}e^{5x} & \frac{d}{dx}e^{-8x} \end{vmatrix} = \begin{vmatrix} e^{5x} & e^{-8x} \\ 5e^{5x} & -8e^{-8x} \end{vmatrix} \\ &= \left(e^{5x}\right)\left(-8e^{-8x}\right) - \left(e^{-8x}\right)\left(5e^{5x}\right) = -8e^{-3x} - 5e^{-3x} = -13e^{-3x}. \end{aligned}$$

We see that the Wronskian is nonzero everywhere (though all we needed was that it be not identically zero), and thus the functions are linearly independent.

This gives a proof of something which seems somewhat clear: we can not get e^{5x} from linear combinations (here meaning constant multiples, since there are only two functions) of e^{-8x}, and vice-versa.

The example we opened the lecture with was $y''' - 2y'' - 15y' = 0$, with characteristic equation $m^3 - 2m^2 - 15m = 0$, which factors to $m(m-5)(m+3) = 0$, giving functions $e^{0x} = 1, e^{5x}, e^{-3x}$. Let us now show that these are linearly independent. Here we need to take two derivatives to construct

[4]In linear algebra, if we have a matrix equation of the form

$$\mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} = \mathbf{0}_{n\times 1},$$

then there are nontrivial solutions $\mathbf{x} \neq \mathbf{0}$ if and only if $\det \mathbf{A} = 0$, which is to say the only solution is $\mathbf{x} = \mathbf{0}$, the trivial one if and only if $\det \mathbf{A} \neq 0$. Here the part of $\mathbf{A}$ is played by the matrix inside the determinant in (10), while the part of $\mathbf{x}$ is played by $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

[5]Here we used the formula for finding the determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

There is also a formula for 3×3 determinants which is not too difficult. A useful visual device is to reproduce the first two columns, placing the copies to the right as below, and then multiply terms on each of the six diagonals, adding the product from the diagonals which go down as we move right, and subtract those which go up:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi.$$

For 4×4 and higher, no such simple device works and linear algebra tricks—such as row reduction or cofactor expansions—need to be used.

the square determinant which is the Wronskian. Below

$$\begin{aligned}
W &= \begin{vmatrix} 1 & e^{5x} & e^{-3x} \\ 0 & 5e^{5x} & -3e^{-3x} \\ 0 & 25e^{5x} & 9e^{-3x} \end{vmatrix} = \left|\begin{matrix} 1 & e^{5x} & e^{-3x} \\ 0 & 5e^{5x} & -3e^{-3x} \\ 0 & 25e^{5x} & 9e^{-3x} \end{matrix}\right| \begin{matrix} 1 & e^{5x} \\ 0 & 5e^{5x} \\ 0 & 25e^{5x} \end{matrix} \\
&= (1)\left(5e^{5x}\right)\left(9e^{-3x}\right) + \left(e^{5x}\right)\left(-3e^{-3x}\right)(0) + \left(e^{-3x}\right)(0)\left(25e^{5x}\right) \\
&\quad - \left(e^{-3x}\right)\left(5e^{5x}\right)(0) - (1)\left(-3e^{-3x}\right)\left(25e^{5x}\right) - \left(e^{5x}\right)(0)\left(9e^{-3x}\right) \\
&= 45e^{2x} + 0 + 0 - 0 + 75e^{2x} - 0 \\
&= 120e^{2x}.
\end{aligned}$$

Once again we see the Wronskian is not identically zero (and in fact, again, is never zero). Thus the functions in the set $\left\{1, e^{5x}, e^{-3x}\right\}$ are linearly independent, so the family of functions given by $y = A + Be^{5x} + Ce^{-3x}$, $A, B, C \in \mathbb{R}$ is a three-dimensional space of functions.

3 An Important Theorem

Now we finally state the theorem which will let us declare with certainty what the solution of a particular LHODE with constant coefficients will be. Here is the theorem:

Theorem 3 *An nth order Linear Homogeneous ODE with constant coefficients will have an n-dimensional solution space. In other words, there exists a linearly independent set of solutions* $\{f_1, f_2, \cdots, f_n\}$ *such that the general solution of the ODE will be an n-parameter family of functions*

$$y = C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x). \tag{11}$$

With what we had before, we can now announce the solutions of the equations we gave at the start of the lecture, and in the examples:

$$\begin{aligned}
y''' - 2y'' - 15y' = 0 \qquad &\iff \qquad y = A + Be^{5x} + Ce^{-3x}, \\
y'' + 3y' - 40y = 0 \qquad &\iff \qquad y = Ae^{5x} + Be^{-8x}, \\
y'' - 3y' + y = 0 \qquad &\iff \qquad y = Ae^{\left(\frac{3+\sqrt{5}}{2}\right)x} + Be^{\left(\frac{3-\sqrt{5}}{2}\right)x}.
\end{aligned}$$

The first requires three linearly independent solutions, and we found three which we proved to be independent, so we had all we needed. The second is order 2, and thus needs exactly two linearly independent solutions, which we found, so we just needed linear combinations of those two functions. For the third equation, rather than using the Wronskian for that particular case we will refer to the first homework problem to show that the functions are indeed independent, and since this equation is of order 2, we are again done.

4 Repeated Roots of the Characteristic Equation

Consider the equation

$$y'' - 6y' + 9y = 0. \tag{12}$$

Using differential operators, we would get

$$\begin{aligned}
&(D^2 - 6D + 9)y = 0 \\
\iff \qquad &[(D-3)(D-3)]y = 0.
\end{aligned}$$

If instead we use the characteristic equation (which we will do more and more of), we would have

$$m^2 - 6m + 9 = 0 \qquad \iff \qquad (m-3)^2 = 0 \qquad \iff m = 3.$$

Our earlier theory tells us two things:

1. e^{3x} is a solution.

2. We need two linearly independent solutions, since our ODE (12) is of order 2.

So we need to find another solution. Here we use an observation which is not at all obvious from what we have done, but is easily proven and has been known for centuries.

Theorem 4 *If we have a linear differential operator L, with constant coefficients, and which annihilates $x^n e^{ax}$, then the operator $(D-a)L$ annihilates $x^{n+1}e^{ax}$.*

Proof: We are assuming L has constant coefficients, so it will commute with $(D-a)$. We also are assuming that $L\left[x^n e^{ax}\right]=0$. This commutivity and the product rule give

$$\begin{aligned}(D-a)L\left[x^{n+1}e^{ax}\right] &= L\left[(D-a)x^{n+1}e^{ax}\right]\\ &= L\left[D(x^{n+1}e^{ax})-ax^{n+1}e^{ax}\right]\\ &= L\left[(n+1)x^n e^{ax}+(x^{n+1})ae^{ax}-ax^{n+1}e^{ax}\right]\\ &= L\left[(n+1)x^n e^{ax}+0\right]\\ &= (n+1)L\left[x^n e^{ax}\right]\\ &= (n+1)\cdot 0\\ &= 0, \qquad \text{q.e.d.}\end{aligned}$$

Going back to our example, the part of L would be played by the second $(D-3)$:

$$(D-3)\underbrace{(D-3)}_{\text{“}L\text{”}}=0.$$

We know that e^{3x} is a solution, but it is annihilated by L and there seems nothing left for the second (leftmost) $(D-3)$. But with this theorem, we get that xe^{3x} is also a solution. We can verify this using the original ODE: $y=xe^{3x}$, $y'=e^{3x}+3xe^{3x}$, $y''=3e^{3x}+3e^{3x}+9xe^{3x}=6e^{3x}+9xe^{3x}$. Plugging these into the (12) would give us

$$\begin{aligned}y=xe^{3x} \implies y''-6y'+9y &= (6e^{3x}+9xe^{3x})-6(e^{3x}+3xe^{3x})+9(xe^{3x})\\ &= e^{3x}(6-6)+xe^{3x}(9-18+9)\\ &= e^{3x}(0)+xe^{3x}(0)\\ &= 0, \qquad \text{q.e.d.}\end{aligned}$$

Using the Wronskian it is easy to show that $\left\{e^{3x}, xe^{3x}\right\}$ is a linearly independent set (see the second homework problem). Since both satisfy the ODE, which is second order so we know the solution set is two-dimensional, we know we have the solution:

$$y=Ae^{3x}+Bxe^{3x}. \tag{13}$$

The theorem is written in a “bootstrap” form easy to prove, but not the easiest to use. For a result which is quick to apply to ODE’s with constant coefficients, we will note the following result, which follows by repeated use of the theorem:

Theorem 5 *Suppose we have an LHODE with constant coefficients, given by*

$$L[y]=0.$$

If $(D-a)^n$ is a factor of L, i.e., if $(m-a)$ is a factor of the characteristic polynomial, then the solution to the LHODE contains the linearly independent set of n functions

$$\left\{e^{ax}, xe^{ax}, x^2e^{ax}, \cdots, x^{n-1}e^{ax}\right\}.$$

We do not prove the independence of the solutions[6] but it should seem reasonable that none of the members of the set above will be linear combinations of the others.

The application of the theorem is very quick, as the following examples show.

Example 3 *Solve the ODE* $y^{(4)} - 8y'' + 16y = 0$.

Solution: The characteristic equation here is

$$\begin{aligned} & m^4 - 8m^2 + 16 = 0 \\ \iff \quad & (m^2 - 4) = 0 \\ \iff \quad & (m-2)^2(m+2)^2 = 0. \end{aligned}$$

We see that $m = 2, 2, -2, -2$ *if we count multiplicity. The solution is thus*

$$y = Ae^{2x} + Bxe^{2x} + Ce^{-2x} + Dxe^{-2x}.$$

Note that the original ODE could be written $(D-2)^2(D+2)^2 y = 0$.

A common mistake in translating the ODE $y^{(4)} - 8y'' + 16y = 0$ into the characteristic equation is to have "m" in place of y, so the $16y$ is erroneously translated into $16m$, when in fact it should be 16 (no m). This is less likely to occur if the ODE is translated into a form using the differential operator $D = \frac{d}{dx}$:

$$(D^4 - 8D^2 + 16)y = 0.$$

Example 4 *Solve the ODE* $y^{(5)} - y^{(4)} = 0$.

Solution: The characteristic equation here is

$$\begin{aligned} & m^5 - m^4 = 0 \\ \iff \quad & m^4(m-1) = 0. \end{aligned}$$

If we count multiplicity, we could say that $m = 0$ *is a solution of multiplicity* 4*, while* $m - 1$ *is a solution of multiplicity* 1*. Now* $m = 0$ *corresponds to* $e^{0x} = 1$ *(a theme which is recurrent), so the solution contains* $1, x, x^2, x^3$ *terms, as well as* e^x *from the other root of the characteristic polynomial. The final solution to the ODE is thus*

$$y = \underbrace{A + Bx + Cx^2 + Ex^3}_{\substack{m=0 \\ \textit{solutions}}} + \underbrace{Fe^x}_{m=1}.$$

(Most texts use capital letters, but we will avoid D for the obvious reason.)

Notice that the equation in Example 4 was order 5 and the solution which presented itself is already a five-parameter family, just as we needed. With a little practice we can "eyeball" the form of the solution and see that the functions are independent (again, we can't get any from linear combinations of the others), and of the correct count, so we must have the solution (assuming our algebra was correct above). In fact, from the procedures we will continue to develop from this section onwards, the correct number of linearly independent solutions will always naturally appear.

[6] This would be very difficult using the Wronskian for the general case, though a purely linear-algebraic method quickly reduces to the independence of $\{1, x, x^2, \cdots, x^{n-1}\}$, which is easy to prove using the Wronskian.

Homework 7-A

1. Show that if $\alpha \neq \beta$, then $\left\{e^{\alpha x}, e^{\beta x}\right\}$ is a linearly independent set of functions.
2. Show that $\{e^{\alpha x}, xe^{\alpha x}\}$ is a linearly independent set of functions.
3. Show that $\left\{1, x, x^2\right\}$ is linearly independent. (If you have already taken linear algebra, you should try to extend this to $\left\{1, x, x^2, x^3\right\}$ and so on. A simple pattern emerges.)
4. Solve the following ODE's using the characteristic equation. (Warning: If not already done so, it is useful to quickly write the equations using differential operators D—without solving the equations that way for now—and to then write the characteristic equation.)
   (a) $(D-4)^5 y = 0$.
   (b) $D^3(D-5)^2 y = 0$.
   (c) $y^{(4)} - 5y''' + y'' = 0$.
   (d) $y^6 - 81y^4 = 0$.
   (e) $y^{(4)} - 6y'' + 9y = 0$.