Lecture 8: LHODEs III

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In this lecture we will take a first look at cases where the characteristic equation has complex, as well as real, solutions. We will delve more deeply into the complex numbers in the next lecture, and then we will be done developing LHODEs.

1 Summary of Earlier Results: What's Missing?

So far we have seen that an LHODE of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1}$$

which can also be written using operators

$$\left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) y = 0, \tag{2}$$

has an associated characteristic polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0. \tag{3}$$

Furthermore, if $a_n \neq 0$, i.e., we truly have an nth order LHODE, then the solution will be an n-parameter (or n-dimensional) family

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_{n-1} y_{n-1} + C_n y_n, \tag{4}$$

where the $y_1, \cdots, y_n$ are linearly independent (which we can check with the Wronskian), and moreover, we can find these functions y_k by looking at the roots (solutions) of the characteristic equation (3) using the following rule:

> If $(m-a)^r$ is a factor of the characteristic polynomial, then include in the y_k's the r functions $e^{ax}, xe^{ax}, x^2 e^{ax}, \cdots, x^{r-1} e^{ax}$.

For a quick example, the LHODE $D^3(D-2)^2(D+3)(D-9)^4 y = 0$, which is order 10 would have characteristic equation

$$m^3(m-2)^2(m+3)(m-9)^4 = 0,$$

and so the solution to the LHODE would be the 9-parameter family

$$y = \underbrace{A + Bx + Cx^2}_{m=0,0,0} + \underbrace{Ee^{2x} + Fxe^{2x}}_{m=2,2} + \underbrace{Ge^{-3x}}_{m=-3} + \underbrace{He^{9x} + Ixe^{9x} + Jx^2 e^{9x} + Kx^3 e^{9x}}_{m=9,9,9,9}.$$

What is missing from this discussion is what to do if the characteristic equation has complex solutions of the form $m = a + bi$, where $i = \sqrt{-1}$ is the **imaginary unit**. For the simplest case this is easy to see:

Theorem 1 *If $m^2 + k^2$, $k \neq 0$ is a factor of the characteristic equation associated with an LHODE, i.e., if $m = ki, -ki$ are solutions of the characteristic equation, then the solution of the LHODE will contain the functions $\sin kx$ and $\cos kx$.*

It is fairly straightforward to see that D^2+k^2 annihilates $\sin kx$ and $\cos kx$:

$$\left(D^2+k^2\right)\sin kx=0,\qquad \left(D^2+k^2\right)\cos kx=0. \tag{5}$$

Note that D^2+k^2 is a second-order operator, and so there must be two independent functions annihilated by this operator. One can check $W(\sin kx,\cos kx)\not\equiv 0$, again assuming $k\neq 0$.

Example 1 *Solve* $y^{(6)}+13y^{(4)}+36y''=0$.

Solution: The equation in operators would look like

$$(D^6+13D^4+36D^2)y=0,$$

which gives us the characteristic equation

$$\begin{aligned} m^6+13m^4+36m^2&=0\\ m^2(m^4+13m^2+36)&=0\\ m^2(m^2+4)(m^2+9)&=0. \end{aligned}$$

This gives

$$\begin{array}{llll} m^2=0, & m^2+4=0, & m^2+9=0, & i.e.,\\ m^2=0, & m^2=-4, & m^2=-9, & i.e.,\\ m=0,0, & m=\pm 2i, & m=\pm 3i. & \end{array}$$

The solution to the characteristic equation is thus $m=0$ *(multiplicity 2),* $m=\pm 2i$, $m=\pm 3i$, *which collectively give the solution to the LHODE:*

$$y=A+Bx+C\sin 2x+E\cos 2x+F\sin 3x+G\cos 3x.$$

The theorem helps when we need to associate functions with $m=\pm bi$ solutions of the characteristic polynomial. But there are other complex solutions besides the purely imaginary ones. We define the set of complex numbers as follows:

$$\mathbb{C}=\left\{a+bi\,\middle|\,a,b\in\mathbb{R}\right\}. \tag{6}$$

So for instance, what to do if we get solutions $m=a\pm bi$?

We should first note that these solutions to a real ($\mathbb{R}$-coefficient) polynomial equation always come in **conjugate pairs**, meaning that if $a+bi$ is a solution, then $a-bi$ is also a solution.[1] Now again it is difficult to see how the operator would factor when we have such solutions of the characteristic equation. Note that if we could factor that part of the operator corresponding to these roots, it would look like

$$\begin{aligned} (D-(a+bi))(D-(a-bi)) &= \left(D^2-(a-bi)D-(a+bi)D+(a+bi)(a-bi)\right)\\ &= \left(D^2-2aD+(a^2+b^2)\right). \end{aligned}$$

[1] This is a simple complex analysis or modern algebra problem. It requires a few steps which we will not need in the main current of this course, but which we list here for the interested reader.

(a) For $z=a+bi\in\mathbb{C}$, define $\bar{z}=a-bi$ to be its **complex conjugate**.

(b) If $z=a+bi,\ z_1=a_1+b_1i,\ z_2=a_2+b_2i\in\mathbb{C}$, then simple checks verify that

- **(i)** $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$;
- **(ii)** $\overline{z_1z_2}=\overline{z_1}\ \overline{z_2}$;
- **(iii)** Repeating the previous result n times, $\overline{z^n}=(\overline{z})^n$;
- **(iv)** $z\in\mathbb{R}\iff\bar{z}=z$;
- **(v)** If $c\in\mathbb{R}$ and $z\in\mathbb{C}$, then $\overline{cz}=c\bar{z}$.

(c) For a polynomial $f(z)=c_nz^n+c_{n-1}z^{n-1}+\cdots+c_1z+c_0$ with real coefficients, i.e., $c_0,c_1,\cdots,c_n\in\mathbb{R}$, if

We are simply not likely to recognize how to factor something like

$$\left(D^2 - 2D + 5\right) = (D - 1 - 2i)(D - 1 + 2i),$$

which is again why we resort to the characteristic equation, which would include the factor m^2-2m+5 corresponding to the operator above. For this particular factor of the characteristic polynomial, we would find it zero when (by the quadratic formula)

$$m = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm \frac{1}{2}\sqrt{-16} = 1 \pm \frac{1}{2}\sqrt{(16)(-1)} = 1 \pm \frac{1}{2} \cdot 4i = 1 \pm 2i,$$

as we would see from the (not so obvious) factoring of the operator $D^2 - 2D + 5$.

For now we will go ahead and state a theorem about general real and complex roots of the characteristic equation, and then see in the next section how all these fit into a coherent theory based on one simple principle.

Theorem 2 *If $m = a + bi$ and $m = a - bi$ are solutions of the characteristic equation, then the solution of the LHODE will include the functions $e^{ax}\sin bx$ and $e^{ax}\cos bx$.*

A short look back at what we had before shows that this sums up all our earlier results, except for those dealing with multiplicities greater than one, which we will address later in this lecture.

2 Euler's Formula

Leonhard Euler[2] derived an equation which unifies the trigonometric and exponential functions through complex analysis. We will not delve into all the rigors of that discipline required to justify Euler's equation, but will give the outline without justifying every step. It is based upon three Taylor (or more precisely, MacLaurin) Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}; \tag{7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(2l)!}; \tag{8}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}. \tag{9}$$

The reason why a series has a **radius of convergence** is that we can replace $x \in \mathbb{R}$ with a complex variable $z = x + iy$, which we represent by a point in the **complex plane** $\mathbb{C}$ (as opposed to the **real**

$f(z_0) = 0$, then $\overline{f(z_0)} = \bar{0} = 0$, so

$$\begin{aligned} 0 &= \overline{f(z_0)} \\ &= \overline{c_n z_0^n + c_{n-1} z_0^{n-1} + \cdots + c_1 z_0 + c_0} \\ &= \overline{c_n z_0^n} + \overline{c_{n-1} z_0^{n-1}} + \cdots + \overline{c_1 z_0} + \overline{c_0} \\ &= c_n \overline{z_0^n} + c_{n-1} \overline{z_0^{n-1}} + \cdots + c_1 \overline{z_0} + c_0 \\ &= f\left(\overline{z_0}\right) \qquad \text{q.e.d.} \end{aligned}$$

The proof is done because essentially what we proved was that if $f(z)$ is a polynomial with real coefficients, then

$$f(z_0) = 0 \implies f\left(\overline{z_0}\right) = 0.$$

Thus if $z_0 = a + bi$ is a solution of the polynomial equation $f(z) = 0$—or any other polynomial equation with real coefficients and real RHS since all can be rewritten into this form—then so is $\overline{z_0} = a - bi$ as was claimed.

[2]Pronounced *Oiler*, a Swiss, 1707–83, and arguably the premier mathematician of his day.

line $\mathbb{R}$) and there a series converges absolutely inside a disk with that radius. This is actually kind of moot for these cases because the radii of convergence are all infinite, i.e., these converge for all values of x real or z complex. We will revisit radii of convergence later in the course when we look at power series solutions to ODE's.

Now suppose we let $\theta \in \mathbb{R}$, and compute $e^{i\theta}$ using its Taylor Series (7) to get a series representation. In doing so we will use the fact that $i = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, etc.

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right). \end{aligned}$$

The real part of the result we can recognize from (8) as $\cos\theta$, and the imaginary part we see from (9) is $i \cdot \sin\theta$. That gives us Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{10}$$

Entire books can be written regarding the significance of this formula, and we will get a good start in this course but really only scratch the surface. Let us consider its implications for an LHODE which we could not solve with our methods from Lectures 6 and 7.

(Note that everything we did above is still valid if we do not assume $\theta \in \mathbb{R}$, but allow also for $\theta \in \mathbb{C}$.)

3 Euler's Formula and LHODE's

Consider the LHODE

$$y'' + 25y = 0. \tag{11}$$

What this really states is that $d^2y/dx^2 + 25y = 0$. What we are about to do would require some time to totally justify, and the required rigor is best left to a complex analysis course, but it is not difficult to see formally what we are doing. The trick is to "complexify" the problem (11), and instead try to solve

$$\frac{d^2Y}{dz^2} + 25Y = 0, \tag{12}$$

where the assumption in (12) is that $Y = Y(z)$, where Y and z are now allowed to be complex variables. We would also assume $D = d/dz$ in the operator form of (12), $(D^2 + 25)y = 0$. (In fact, when we only measure change in the real direction, $d/dz = d/dx$, but d/dz measures in all directions, analogous to the gradient.) If we do so, we still get the characteristic equation

$$m^2 + 25 = 0 \iff m^2 = -25 \iff m = \pm 5i.$$

Now formally that would give us

$$Y(z) = Ae^{5iz} + Be^{-5iz}, \tag{13}$$

in the spirit of the previous lectures.[3] Of course since this is a complexified problem, we have to go all the way and allow $A, B \in \mathbb{C}$ as well. Now from Euler's Formula we have

$$\begin{aligned} Y(z) &= A(\cos 5z + i\sin 5z) + B(\cos(-5z) + i\sin(-5z)) \\ &= A(\cos 5z + i\sin 5z) + B(\cos 5z - i\sin 5z) \\ &= (A + B)\cos 5z + (Ai - Bi)\sin 5z. \end{aligned} \tag{14}$$

[3] which stated that if $m = a$ is a solution of the characteristic equation, then $y = e^{ax}$ is a solution of the LHODE.

We used here that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ (which, incidentally, follow quickly from their series (9) and (8)). Now we will wave our hands, avoiding some rather messy and distracting details, to argue that we can now get the actual solution of the original, real LHODE (11):

1. Our LHODE was not complex at the beginning, so we expect the actual answer to be real.
2. The answer should be two-dimensional, i.e., since the LHODE was order 2, the solution should be a two-parameter family, and be the span of two linearly independent functions.
3. If we look at (14), with A and B being complex, we can choose $A, B \in \mathbb{C}$ to get $M \cos 5x + N \sin 5x$, for any $M, N \in \mathbb{R}$ we would like.[4]
4. If we then constrain z to the real line, we get the *real* solution we desire:
$$y(x) = M \cos 5x + N \sin 5x. \tag{15}$$
5. A quick check shows that
   (a) this is a solution of the LHODE (11);
   (b) the two functions are linearly independent (and actually have a nice Wronskian);
   (c) we have exactly the dimension of solution that the given (2nd order) LHODE required.

   Thus (15) must be *the* solution to the LHODE.

Once the upshot of the method is digested, it can be shortened considerably: we know we can complexify the problem, get some sines and cosines, and these will work with the original problem as well.

Example 2 *Solve the LHODE $y^{(4)} + 5y'' = 0$.*

Solution: It is still useful to write out the operator version of the LHODE before leaping to the characteristic equation:

$$(D^4 + 5D^2)y = 0.$$

The characteristic equation is then

$$m^4 + 5m^2 = 0 \iff m^2(m^2 + 5) = 0 \iff m^2 = 0,\ m^2 = -5.$$

The first part gives us $m = 0$ (multiplicity 2), and the other gives the conjugate pair $m = \pm i\sqrt{5}$ with multiplicity 1. The first ($m = 0$) root gives us the terms $A + Bx$, and the imaginary roots collectively give us the terms $E \cos\left(\sqrt{5}x\right) + F \sin\left(\sqrt{5}x\right)$. The solution to the LHODE is then the sum of these:

$$y = A + Bx + E \cos\left(\sqrt{5}x\right) + F \sin\left(\sqrt{5}x\right).$$

[4] This is a linear algebra problem. Since $A, B \in \mathbb{C}$, we can write $A = a_1 + a_2 i$, $B = b_1 + b_2 i$, and then

$$(A + B) \cos 5z + (Ai - Bi) \sin 5z = M \cos 5z + N \sin 5z$$

becomes a system of linear equations in a_1, a_2, b_1, b_2, for any given (fixed) $M, N \in \mathbb{R}$. The system is always solvable but we will not bother here to give details. (The basic idea is that the real terms have to match, the imaginary terms have to match, the cosine terms have to match, and the sine terms have to match. From these you set up your system of four equations in four unknowns, a_1, a_2, b_1, b_2.)

4 General Real and Complex Roots

Now that we know why we get Theorem 1 for the pure imaginary solutions of the characteristic equation, we will again look at more general complex solutions. The same arguments apply there.

Suppose $m = a \pm bi$ are solutions of the characteristic equation. For the complexified case, we would then formally get

$$\begin{aligned} Y(z) &= Ae^{(a+bi)z} + Be^{(a-bi)z} \\ &= Ae^{az}e^{ibz} + Be^{az}e^{-ibz} \\ &= Ae^{az}(\cos bz + i\sin bz) + Be^{az}(\cos(-bz) + i\sin(-bz)) \\ &= Ae^{az}(\cos bz + i\sin bz) + Be^{az}(\cos bz - i\sin bz) \\ &= (A+B)e^{az}\cos bz + (Ai - Bi)e^{az}(\sin bz). \end{aligned}$$

As before, when we pull back to the real case, we extract the solutions

$$e^{ax}\cos bx, \qquad e^{ax}\sin bx. \tag{16}$$

This is consistent with Theorem 2. Also note that since $\cos 0 = 1$ and $\sin 0 = 0$, the real case $m = a$ ($b = 0$) is contained here. Also contained here (left to the reader to see why) is the pure imaginary case $m = \pm bi$.

5 Putting it all Together

The only detail left is to include the case where complex roots show up with higher multiplicities than 1. If we take what we had before regarding multiplicity in real roots, and formally have as our solution e^{ax} for a real *or* complex, and make the adjustments for the complex case from the section immediately preceding, we get a very general set of rules (actually the first two contained in the last, as before!):

1. If $m = a$ ($a \in \mathbb{R}$) is a solution with multiplicity r of the characteristic equation, then the solution of the LHODE must contain the r, linearly independent functions $e^{ax}, xe^{ax}, x^2e^{ax}, \cdots, x^{r-1}e^{ax}$.

2. If $m = \pm bi$ are conjugate pairs, which are solutions of multiplicity r of the characteristic equation, then the solution of the LHODE must contain the $2r$, linearly independent solutions $\sin bx,\ \cos bx,\ x\sin bx,\ x\cos bx,\ x^2\sin bx,\ x^2\cos bx,\ \cdots,\ x^{r-1}\sin bx,\ x^{r-1}\cos bx$.

3. If $m = a \pm bi$ are conjugate pairs, which are solutions of multiplicity r (so $a + bi$ and $a - bi$ each have multiplicity r) of the characteristic equation, then the solution of the LHODE must contain the $2r$, linearly independent solutions $e^{ax}\sin bx,\ e^{ax}\cos bx,\ xe^{ax}\sin bx,\ xe^{ax}\cos bx,\ x^2e^{ax}\sin bx,\ x^2e^{ax}\cos bx,\ \cdots,\ x^{r-1}e^{ax}\sin bx,\ x^{r-1}e^{ax}\cos bx$.

Notice that in all these cases, when the root of the characteristic polynomial shows up with multiplicity greater than 1, we simple include increasing powers of x, multiplying the more obviously annihilated functions (i.e., those annihilated by the first instance of that factor in the operator), until we get enough functions to account for the order of that corresponding part of the operator or, equivalently, the multiplicities of the roots of the characteristic polynomial.

Example 3 *Solve the LHODE* $y^{(4)} + 18y'' + 81y = 0$.

Solution: The operator version of the problem is

$$(D^4 + 18D^2 + 81)y = 0.$$

The characteristic equation is

$$m^4 + 18m^2 + 81 = 0 \iff (m^2+9)^2 = 0 \iff m^2 = -9 \text{ (twice!)}.$$

We get $m = \pm 3i$, with multiplicity 2 each (or we could say the conjugate pair has multiplicity 2). The solution will be

$$y = A \sin 3x + B \cos 3x + Fx \sin 3x + Gx \cos 3x.$$

Example 4 *Solve $(D^2 + 2D + 3)^2 (D - 5)^3 y = 0$.*
<u>Solution</u>*: The characteristic equation here is*

$$(m^2 + 2m + 3)^2 (m - 5)^3 = 0.$$

Now the $(m - 5)^3$ term gives us $m = 5$ (multiplicity 3). The other term requires the quadratic equation:

$$m = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \frac{1}{2} \cdot \sqrt{-8} = -1 \pm \frac{1}{2} 2\sqrt{2} i = -1 \pm i\sqrt{2}.$$

These correspond to the functions $e^{-1x} \sin \sqrt{2}x$ and $e^{-1x} \cos \sqrt{2}x$. However, each of these occurs with multiplicity 2. The general solution of the LHODE is thus

$$y = Ae^{5x} + Bxe^{5x} + Cx^2 e^{5x} + Ee^{-x} \sin \sqrt{2}x + Fe^{-x} \cos \sqrt{2}x + Gxe^{-x} \sin \sqrt{2}x + He^{-x} \cos \sqrt{2}x.$$

<u>Homework 8-A</u>

1. Use Euler's equation to show that
   (a) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ (also known as $\cosh ix$),
   (b) $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ (also known as $\frac{1}{i} \sinh ix$).
2. Solve $y^{(8)} + 8y^{(6)} + 16y^{(4)} = 0$. (Be sure to always write the operator form, and factor, factor!)
3. Solve $y^{(4)} + 4y''' + 13y'' = 0$.
4. Solve $y^{(4)} - y = 0$.
5. Solve $D^5 (D + 2)^3 (D^2 + 5)^3 (D^2 - 6D + 10)^2 y = 0$.