Lecture 9: LHODEs IV

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1 The Development So Far

At this point we know that, given an LHODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0, \tag{1}$$

with the operator version and characteristic equation given respectively by

$$\big(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\big) = 0, \tag{2}$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0, \tag{3}$$

that knowing the solutions of (3) gives us the general solution of the original LHODE, according to the following rules (see Section 5 of Lecture 8):

1. If $m = a$ is a real solution with multiplicity r of the characteristic equation, then the solution of the LHODE must contain the r, linearly independent functions e^{ax}, xe^{ax}, $x^2 e^{ax}$, $\cdots$, $x^{r-1} e^{ax}$.

2. If $m = a \pm bi$ are complex solutions with multiplicity r of the characteristic equation, then the solution of the LHODE must contain the $2r$, linearly independent functions $e^{ax} \sin bx$, $e^{ax} \cos bx$, $xe^{ax} \sin bx$, $xe^{ax} \cos bx$, $\cdots$, $x^{r-1} e^{ax} \sin bx$, $x^{r-1} e^{ax} \cos bx$.

   If $a = 0$ (so the solutions are of form $\pm bi$), then we need $\sin bx$, $\cos bx$, $x \sin bx$, $x \cos bx$, $\cdots$, $x^{r-1} \sin bx$, $x^{r-1} \cos bx$.

Example 1 *In Homework 8A, #5 we were asked to solve the LHODE*

$$D^5 (D+2)^3 (D^2+5)^3 (D^2 - 6D + 10)^2 y = 0.$$

The characteristic equation is

$$m^5 (m+2)^3 (m^2+5)^3 (m^2 - 6m + 10)^2 = 0.$$

The solutions are

$$\begin{aligned}
m &= 0 \text{ (multiplicity 5)}, \\
m &= -2 \text{ (multiplicity 3)}, \\
m &= \pm\sqrt{5} i \text{ (multiplicity 3)}, \\
m &= \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i \text{ (multiplicity 2)}.
\end{aligned}$$

Putting all these together gives us

$$\begin{aligned}
y = {} & \underbrace{A + Bx + Cx^2 + Ex^3 + Fx^4}_{m=0 \ (mult.\ 5)} + \underbrace{Ge^{-2x} + Hxe^{-2x} + Ix^2 e^{-2x}}_{m=-2 \ (mult.\ 3)} \\
& + \underbrace{J \sin\sqrt{5}x + K \cos\sqrt{5}x + Lx \sin\sqrt{5}x + Mx \cos\sqrt{5}x + Nx^2 \sin\sqrt{5}x + Ox^2 \cos\sqrt{5}x}_{m=\pm\sqrt{5}i \ (mult.\ 3\ each)} \\
& + \underbrace{Pe^{3x} \sin x + Qe^{3x} \cos x + Rxe^{3x} \sin x + Sxe^{3x} \cos x}_{m=3\pm i \ (mult.\ 2\ each)}.
\end{aligned}$$

What is left to do now is to solve those equations with characteristic polynomials which are not already factored, using more sophisticated algebraic methods than we used previously.

For instance, we would like to be able to solve equations such as

$$y^{(8)} + y = 0, \tag{4}$$

which gives the characteristic equation (think $(D^8 + 1)y = 0$) $m^8 + 1 = 0$, or $m^8 = -1$. Thus we need to be able to find all (eight) of the 8th roots of -1. This is a complex variable problem.

For another example, we may have an equation like

$$y''' - 2y'' - 5y' + 6y = 0, \tag{5}$$

which gives the characteristic equation $m^3 - 2m^2 - 5m + 6 = 0$, which does factor $(m - 3)(m + 2)(m - 1) = 0$, but how to guess that? This is a simpler, college algebra-type problem, but the method bears repeating. This method, however, we will cover in class, though the idea is to look for factors $m = p/q$, where $p \in \{\pm 1\}$ and $q \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$, the integer factors of 1 (coefficient of m^3) and 6 (coefficient of m^0), respectively. Once we have one factor (like $m = -1$) we use long division (dividing here by $(m + 1)$) to be left with a quadratic we can factor or use the quadratic formula to find the other roots.

In this lecture we will explore the complex variables problem, by introducing the *polar form* of a complex variable, and then use it to find these roots.

2 Polar Form of a Complex Number

If we have a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, we define its length (a.k.a. absolute value, magnitude, modulus) $|z|$ by

$$|z| = \sqrt{a^2 + b^2}. \tag{6}$$

We also define the real and imaginary parts of z to be

$$\text{Re}(z) = a, \qquad \text{Im}(z) = b. \tag{7}$$

Thus we could also write $|z| = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$, and $z = \text{Re}(z) + i\text{Im}(z)$.

In Figure 1 we draw z, now an element of the complex plane $\mathbb{C}$, just as we would draw the ordered pair (a, b) in the usual Cartesian Plane $\mathbb{R}^2$. (Note that z is often drawn as a vector, and indeed, complex numbers do "add as vectors.")

The complex number z will form an angle with the origin as vertex, and the positive x-axis (a.k.a. *real axis*), which we will call θ, measured counter-clockwise as in trigonometry and usual polar coordinates. In complex variables, the angle θ is also called the **argument** of z, and we write

$$\arg z = \theta. \tag{8}$$

If, in the spirit of polar coordinates, we take $r = |z|$, then we get

$$\text{Re}(z) = r\cos\theta, \qquad \text{Im}(z) = r\sin\theta \tag{9}$$

and so

$$z = r\cos\theta + ir\sin\theta. \tag{10}$$

Recall Euler's formula, that $e^{i\theta} = \cos\theta + i\sin\theta$. Since we can factor z into $z = r(\cos\theta + i\sin\theta)$, we see that this quickly becomes

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{11}$$

Equation 11 is called the **polar form** of z. Basically we identify z with its distance r from the origin, and the angle it makes with the origin and the real axis. In that sense this is in the spirit of the usual polar coordinates.

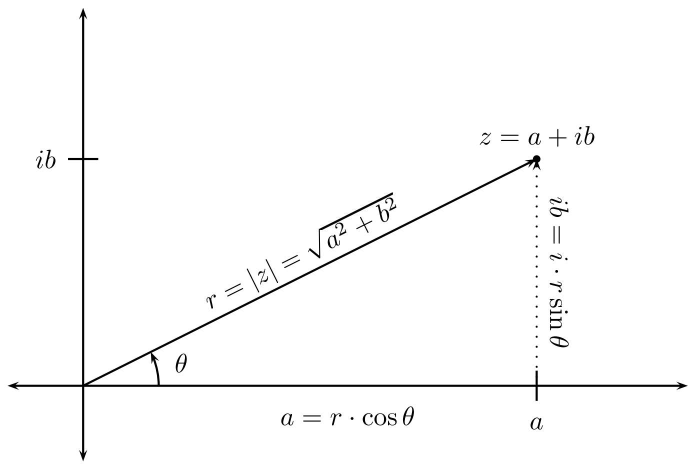

Figure 1: A complex number $z = a + ib$ written in polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

Finding θ requires the usual tricks of trigonometry. It is important to know what quadrant (or axis) z lies in, and then we can use the fact that

$$\tan\theta = \frac{b}{a} = \frac{\text{Im}(z)}{\text{Re}(z)}. \tag{12}$$

Now θ is not unique. Indeed, if we find an angle θ, then any angle $\theta + n(2\pi)$ also works, for any integer n. This is sometimes considered a drawback of polar coordinates, but here we will actually exploit it.

Example 2 *Give three polar representations of $z = 1 - i$.*
<u>Solution</u>: First we calculate r: $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. To find θ, we notice that z lies in the fourth quadrant, and that $\tan\theta = \frac{-1}{1}$, which gives the reference angle ($\tan^{-1} 1$) to be $\pi/4$. Angles θ which work are $\theta = -\pi/4 + n(2\pi)$, so we can write

$$\begin{aligned} z = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right) &= \sqrt{2}\exp\left(i \cdot \frac{-\pi}{4}\right), \\ = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) &= \sqrt{2}\exp\left(i\frac{7\pi}{4}\right) \\ = \sqrt{2}\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right) &= \sqrt{2}\exp\left(i\frac{15\pi}{4}\right). \end{aligned}$$

Example 3 *Give four polar representations of $2i$.*
<u>Solution</u>: (The obvious picture should be drawn or imagined.) Note that $\tan\theta$ is undefined for such z, since $\theta = \pi/2$ is one candidate. Adding 2π, 4π (as we did in the previous example) and 6π to this θ gives us three other possibilities. $|2i| = \sqrt{0^2 + 2^2} = 2$, which is also obvious from a drawing. Four polar representations of $2i$ are given below:

$$2i = 2\exp\left(i \cdot \frac{\pi}{2}\right) = 2\exp\left(i \cdot \frac{5\pi}{2}\right) = 2\exp\left(i \cdot \frac{9\pi}{2}\right) = 2\exp\left(i \cdot \frac{13\pi}{2}\right).$$

3 Multiplication in Polar Form

Something very nice happens when we multiply two complex numbers, and it is clear when we use the polar form. Suppose $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then simple algebra (which incidentally, still holds true in $\mathbb{C}$, though we will not prove all the details here), gives us

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{13}$$

Thus when we multiply z_1 and z_2, the moduli multiply, but also the arguments (angles) combine in a sum. Indeed, from (13), we get

$$|z_1 z_2| = |z_1| \cdot |z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2). \tag{14}$$

This is one reason why complex numbers are useful in modelling physical phenomena, since a dilation by ρ and a rotation by an angle α can be represented as a multiplication by a single, complex scalar $\rho e^{i\alpha}$.

If we apply (14) n times, we get De Moivre's[1] formula (below), which can be written several ways. Here we assume $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, and write

$$z^n = r^n e^{in\theta}, \qquad \text{i.e.,} \tag{15}$$

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta). \tag{16}$$

4 Finding nth Roots

A kind of converse to De Moivre's formula can be seen from the simple calculation

$$\begin{aligned}\left[r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\right]^n &= \left(r^{1/n}\right)^n\left(\cos\left\{n\cdot\frac{\theta}{n}\right\} + i\sin\left\{n\cdot\frac{\theta}{n}\right\}\right)^n \\ &= r(\cos\theta + i\sin\theta).\end{aligned}$$

If we use exponential notation this is easier to see:

$$\left(r^{1/n}e^{i\theta/n}\right)^n = re^{i\theta}.$$

The upshot of this is that $r^{1/n}e^{i\theta/n}$ is an nth root of $re^{i\theta}$.

It is a well-known fact that any nonzero complex number will have n distinct nth roots for any positive integer n.[2] The ambiguity of the actual measure of θ can be exploited here to find all the roots. If we pick n consecutive choices for θ and calculate $r^{1/n}e^{i\theta/n}$ for each, then we will get our n roots. (Any more and we will repeat the roots we have.)

Example 4 *Find the four fourth roots of $z = -16$.*

Solution: First we note that $z = 16e^{i\pi}$ which we can readily see from a drawing of $z = -16$ in the plane. Since there is no single fourth root of this z, we will use quotes as we calculate these. Here is how we will proceed. First calculate $|z| = \sqrt{0^2 + (-16)^2} = 16$, which is obvious from a drawing as well. Next, write

$$z = 16e^{i\pi},\ 16e^{i\cdot 3\pi},\ 16e^{i\cdot 5\pi},\ 16e^{i\cdot 7\pi}.$$

[1] Abraham de Moivre, 1667–1754. Known to be a quiet, hard working and unassuming mathematician, he traveled to England and became a close friend of Sir Isaac Newton after reading Newton's *Principia Mathematica* and realizing how far behind his otherwise modern studies were rendered by that work. His papers are described as few but interesting.

[2] We will have nearly all the pieces to prove this, but the actual counting argument we will leave to the interested reader.

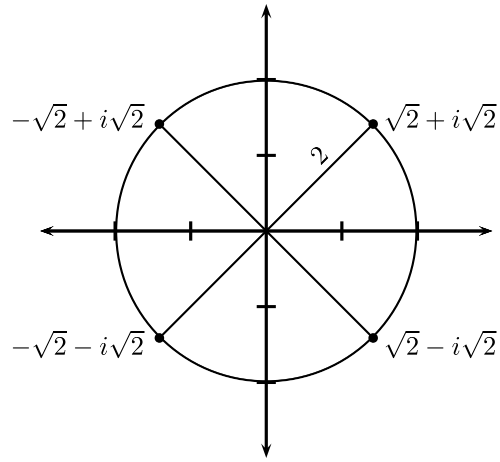

Figure 2: "Spoked wheel" formed by the four fourth roots of -16, i.e., $\pm\sqrt{2}\pm i\sqrt{2}=\sqrt{2}(\pm 1\pm i)$. Notice that any of these rotations from the positive real axis, if repeated four times, and the moduli $\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=2$, raised to the fourth power, will bring us back to -16 as desired.

Next we will "raise to the (1/4)th power" each of these.

$$\text{“}z^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right),$$
$$\text{“}z^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{3\pi}{4}} = 2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) = 2\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right),$$
$$\text{“}z^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{5\pi}{4}} = 2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right) = 2\left(\frac{-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right),$$
$$\text{“}z^{1/4}\text{”} = 16^{1/4}e^{i\cdot\frac{7\pi}{4}} = 2\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right),$$

The four fourth roots of -16 *are thus* $\sqrt{2}+\sqrt{2}i$, $-\sqrt{2}+\sqrt{2}i$, $-\sqrt{2}-\sqrt{2}i$, $\sqrt{2}-\sqrt{2}i$, *or collectively,* $\pm\sqrt{2}\pm\sqrt{2}i$.

Why do we care about such things? The simple answer is, if we have the LHODE

$$(D^4+16)y=0,$$

then the characteristic equation is $m^4+16=0$, or $m^4=-16$, so we do have occasion to need the four fourth roots of a number like -16. Note that the solution to the LHODE would be

$$y = \underbrace{Ae^{\sqrt{2}x}\sin\sqrt{2}x + Be^{\sqrt{2}x}\cos\sqrt{2}x}_{m=\sqrt{2}\pm\sqrt{2}i} + \underbrace{Ee^{-\sqrt{2}x}\sin\sqrt{2}x + Fe^{-\sqrt{2}x}\cos\sqrt{2}x.}_{m=-\sqrt{2}\pm\sqrt{2}i}$$

Notice that the roots will form angles of $2\pi/4=\pi/2$ with each other in the sense of Figure 2. In fact, all the nth roots always form a "spoked wheel" with angles $2\pi/n$, as the process produces.

Homework 9-A

1. Find and graph all the cube roots of $z = 27$. Then use these to solve the LHODE $y''' - 27y = 0$. (One can do this with the quadratic equation, but use the method of Example 4.)

2. Find and graph all the fourth roots of $z = -1$. Then use these to solve $y^{(4)} + y = 0$.