Lecture 12: Solutions by Series

March 4, 2007

Here we look at methods of solving ODE's using series. The basic idea is that we assume the solution has a power series expansion, and use the equation to find the coefficients of the series. We begin with an example for which we know the solution, then proceed to list some of the theory of Taylor and power series that will be relevant, and finally apply the technique to problems for which we would be hard-pressed, or unable, to find solutions with previous methods.

1 An Example

Example 1 *Suppose we wish to find a series solution*

$$y = \sum_{k=0}^{\infty} a_k x^k, \tag{1}$$

centered at $x = 0$*, for the equation* $y'' + y = 0$*. We do this "plugging" the series into the equation. Before doing so we note that*

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k x^k \\ \implies y' &= \sum_{k=1}^{\infty} a_k k x^{k-1} \\ \implies y'' &= \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}. \end{aligned}$$

If we just plug these into our equation (1)*, we then have*

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0. \tag{2}$$

A fact from power series theory that we will need now, and will list again later, is that the coefficients of any two power series for the same function, centered at the same point, must be the same. For our purposes that means

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k \implies \begin{cases} a_0 = b_0 \\ a_1 = b_1 \\ a_2 = b_2 \\ \quad \vdots \end{cases}.$$

Note that this says that the coefficient of 1 *is the same in both series, the coeffient of* x *is the same, the coefficient of* x^2 *is the same, and so on. Morevoer, the only correct power series expansion of the RHS of* (2)*is one in which all coeffcents are zero; that expansion would obviously work, so it must be the only one.*

Now we employ a device for "adjusting" the coefficients in a sum, very carefully. For a simple example, consider that the following three series are the same, a fact that is clear if we write out the first few terms of each:

$$\sum_{i=0}^{\infty} c_{i+1} = \sum_{i=1}^{\infty} c_i = \sum_{i=2}^{\infty} c_{i-1}.$$

If we adjust upwards our starting number for our index i*, we have to adjust downward the index inside our formula for our summed terms; if we adjust downward our starting point, then we adjust upward the index in our summand. With that in mind, we realize that both summations in our ODE* (2) *begin with an* $x^0 = 1$ *term, then an* $x^1 = x$ *term, and so on. Thus we can rewrite* (2) *to read*

$$\begin{aligned} &\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0 \\ \Longrightarrow &\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}\, x^k + \sum_{k=0}^{\infty} a_k x^k = 0 \end{aligned}$$

$$\Longrightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} + a_k\right] x^k = 0. \tag{3}$$

Since the RHS is zero (and has unique power series expansion $\sum 0x^k$ *centered at* $x = 0$*), we can state a relationship among the coefficients:* $(k+2)(k+1)a_{k+2} + a_k = 0$*, i.e.,*

$$a_{k+2} = \frac{-a_k}{(k+2)(k+1)}. \tag{4}$$

This is a recursion relation, where "later" terms are determined by the earlier terms. Notice that it does not tell us **exactly** *what the coefficients are. In fact it should not, because we expect to have a 2-parameter family of curves as our solution. Indeed, we will use* (4) *twice: once where we let* a_0 *be a parameter, and the other where we let* a_1 *be a parameter.*

$$\begin{aligned} a_0 &= a_0 \\ a_2 &= \frac{-a_0}{(2)(1)} \\ a_4 &= \frac{-a_2}{(4)(3)} = \frac{a_0}{4!} \\ a_6 &= \frac{-a_4}{(6)(5)} = -\frac{a_0}{6!} \\ &\vdots \end{aligned} \qquad\qquad \begin{aligned} a_1 &= a_1 \\ a_3 &= \frac{-a_1}{(3)(2)} \\ a_5 &= \frac{-a_3}{(5)(4)} = \frac{a_1}{5!} \\ a_7 &= \frac{-a_5}{(7)(6)} = -\frac{a_1}{7!} \\ &\vdots \end{aligned}$$

At this point we declare that patterns have been established, and we write our solution as

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k x^k \\ &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \end{aligned}$$

Note that this is exactly the same as writing $y = a_0 \cos x + a_1 \sin x$*, if we use the Taylor/MacLaurin power series for* $\sin x$ *and* $\cos x$*.*

For the most part, we will use the same basic outline as used above. We might also notice that the above form is particularly useful for initial value problems (IVP's) if the data is given $y(0) = a_0$, $y'(0) = a_1$. Otherwise it can be rather difficult to find a_0 and a_1 if we do not know the two independent solutions except in their power series representations.

2 Review of Taylor and Power Series

Here we offer a quick review of some facts regarding Taylor series:[1]

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad x \in \mathbb{R} \tag{5}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad x \in \mathbb{R} \tag{6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \qquad x \in \mathbb{R} \tag{7}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1, \tag{8}$$

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}, \qquad |x-a| < R. \tag{9}$$

In fact (8) converges at $x = -1$ as well (alternating series test). The general formula for the Taylor Series is (9), and there will be an interval (perhaps one point, perhaps $R = \infty$), in which a function which can be differentiated as many times as we like, will have its Taylor Series converging around the center of the series.

Taylor Series are closely related to general Power Series, which are always centered at some point $x = c$. There is much use for functions *defined* by a power series:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k. \tag{10}$$

It can be shown that the series on the RHS of (10) is in fact the Taylor Series for $f(x)$ centered at $x = a$, so Taylor and Power Series are essentially the same, where defined.

It should also be pointed out that not all functions can be represented by a convergent Taylor Series, and those that can often need a different center to find a series for the function that converges. For instance, if we want a Taylor Series for $\tan x$, it will need to have a different center for x near, say, $\pi/4$ than for x near $3\pi/4$, since there is a vertical asymptote between them (at $\pi/2$). Indeed, the fact that a series only converges within some radius around the series is significant. Also within such a radius the power series will be continuous.

In fact it is, in math-speak, more than continuous. The adjective for a function which can be written by a power series within some interval is *real-analytic*. The function $1/(1-x)$ is real-analytic in any open interval not containing $x = 1$, though even in $(1, \infty)$ there is not a single series

[1] Some quick research reveals Brook Taylor, English mathematician, 1625–1731, as perhaps first to give a general method and proof for finding the series that bear his name, in 1715, though some aspects of the theory and results were known for some time. In particular, James Gregory (1638–1675) calculated several such series, and Johann Bernoulli (Swiss, 1667–1748, tutored by L'Hôpital) laid claim to priority, though this is in much dispute. Even earlier, Madhava of Kerala, India (1350–1425) is given credit for many well-known (and not-so-well-known) series results, though his original works have largely perished and we rely upon later scholars' accounts. The Scottish mathematician Colin Maclaurin (1698–1746) published what is essentially the Taylor Series centered at zero, in *Treatise of Fluxions*, 1742, apparently unaware of Taylor's earlier work.

representation that works for the whole interval as, for instance, the series centered at $c = 2$ (radius 1) will not work at $x = 5$, but the series centered at $x = 6$ will (radius 5). Some shortcuts for finding the radius of convergence without knowing the actual coefficients will be discussed later.

Already mentioned is that such a series is unique:

$$\left.\begin{array}{r} f(x) = \displaystyle\sum_{k=0}^{\infty} a_k(x-c) \\[2em] f(x) = \displaystyle\sum_{k=0}^{\infty} b_k(x-c)^k \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} a_0 = b_0 \\ a_1 = b_1 \\ a_2 = b_2 \\ \quad\vdots \end{array}\right. . \tag{11}$$

So indeed the power series represenation for $f(x)$ must be the same as the Taylor Series (with the part of c played by a):

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k \qquad \Longrightarrow \qquad a_k = \frac{f^{(k)}(a)}{k!}. \tag{12}$$

This has algebraic implications as well. So for instance, the Taylor Series for e^{2x} and e^{x^2} must be, respectively,

$$\begin{aligned} e^{2x} &= \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots \\ e^{x^2} &= \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots . \end{aligned}$$

This is because these clearly work point-wise for any x (because, respectively, $2x, x^2 \in \mathbb{R}$) so these must be legitimate series, both centered at $x = 0$, so they must be *the* Taylor Series of their respective functions.

Similarly there are some theorems from calculus:

Theorem 1 Manipulations with Power Series: *Suppose we are given a function defined by a power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \tag{13}$$

which converges[2] *in some open interval $|x-a| < R$, where $R > 0$. Then inside that same interval,*

1. *all derivatives of f exist and are continuous;*
2. $f^{(n)}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{d^n}{dx^n}\left[a_n(x-a)^n\right];$
3. $\displaystyle\int_a^x f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{(n+1)!}$

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[2] In fact, the convergence is *absolute* within the open interval $|x-a| < R$. In other words,

$$\sum_{n=0}^{\infty} |a_n(x-a)^n| < \infty,$$

so the convergence does not rely on cancellation from alternation, for example. A theorem states that any absolutely (as in absolute values) convergent series also converges in the normal sense, that the partial sums for a convergent sequence.

What this theorem really states is that we can differentiate and integrate term-by-term. We can sometimes combine these with previous facts to generate Taylor Series without the laborious process of actually computing the coefficients using $a_k = f^{(k)}(a)/k!$. Consider the following:

$$\begin{aligned}
\tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\,dt \\
&= \int_0^x \frac{1}{1-(-t^2)}\,dt \\
&= \int_0^x \left[1 - t^2 + t^4 - t^6 + t^8 - \cdots\right] dt \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .
\end{aligned}$$

The third line in the calculation above replaces x with $-t^2$ in the series for $1/(1-x)$ in (8). Note that we can also use summation notation. Indeed, a check gives us

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}\right] = \sum_{k=0}^{\infty}\left[(-1)^k x^{2k}\right] = \sum_{k=0}^{\infty}(-x^2)^k,$$

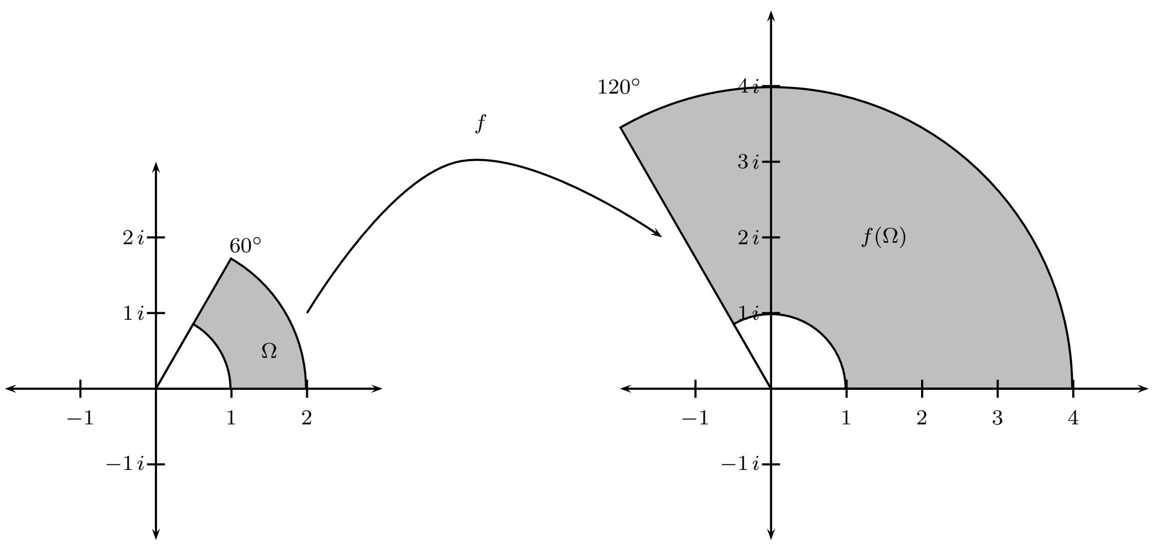
which yields $1 - x^2 + x^4 - x^6 + \cdots = 1/(1-(-x^2)) = 1/(1+x^2)$, the derivative of $\tan^{-1} x$. Here and previously we used the formula for the sum of a geometric series.[3]

3 Some Complex Variables Theory

In fact some insight into real-analytic functions can be gained by looking to the complex plane $\mathbb{C}$. Functions $f(z) = u(z) + iv(z)$, i.e., functions f which input and output complex numbers, are difficult to visualize in the usual way, but have so much structure that makes certain real-variable problems quite simple.[4]

[3] Recall the geometric seris $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$, assuming $|r| < 1$. If $a \neq 0$ and $|r| \geq 1$, then the series diverges.

[4] The usual way to visualize the action of a complex function is to look at the image of its domain, or portions thereof, within the range. For instance, the functions $f(z) = z^2$ will take a number $z = re^{i\theta}$ and return $z^2 = r^2 e^{2i\theta}$. So the region Ω given below and left is "mapped" to the region $f(\Omega)$ below right, under this mapping f:



A function being complex-analytic also means that it can be written as a (absolutely) convergent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k,$$

but where $a, a_k, z, f(z)$ are all allowed to be complex. Absolutely convergent means what it does in $\mathbb{R}$-valued series, except $|z| = \sqrt{x^2+y^2}$ if $z = x + iy$. In the theory of $\mathbb{C}$, it is shown that a complex-valued function f with complex derivative at z_0, i.e., so that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

exists, where $\Delta z \to 0$ is allowed to occur along *any* direction in the complex plane (not just the two "real" directions), in fact $f(z)$ will have *all* derivatives existing there, i.e., $f'(z_0)$ exists $\implies$ $f^{(n)}(z_0)$ exists for all $n = 1, 2, 3, \ldots$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists } \implies f''(z_0), f'''(z_0), f^{(4)}(z_0), \ldots \text{ also exist.}$$

The proof is inductive: $f^{(n)}(z_0)$ exists $\implies$ $f^{(n+1)}(z_0)$ exists. This is not true in $\mathbb{R}$.[5]

What is good to know—though the development really should be done in a complex analysis course—is that many of our "nicer" real-valued functions are also complex-analytic functions in $\mathbb{C}$: polynomials, exponentials, roots (though which "branch" of, say, the square root, must be specified), sine and cosine, and even logarithms (which also require some "branch" and have to avoid $z = 0$).

What is also useful is that a function whose derivative in $\mathbb{C}$ exists in an open set $\Omega \subseteq \mathbb{C}$ will be complex-analytic there, and have a Taylor series converge in any disk within Ω. Thus we can see

$$\frac{1}{1+z^2} = \frac{1}{1-((-1)z^2)} = \sum_{k=0}^{\infty} (-1)^k z^{2k} = \sum_{k=0}^{\infty} \left((-1)^k (z-0)^{2k}\right) \tag{14}$$

converges until we stray the distance 1 from the center 0, because in the complex plane we have the function undefined at $z = \pm i$, which are 1 unit away from the center of the series. Hence the real-valued series where we replace z with x above also converges until we stray a distance 1 from the origin, so convergence truly occurs within a "radius," even if our original problem is real and not outwardly complex. The result is useful here, though the development belongs to a complex variables course.

Similarly, the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ we an easily suspect will not converge more than one unit from the center 0, because 1 unit right of 0 is a discontinuity (in the form of a vertical asymptote). If we believe there is a definite distance (radius of convergence) for such a series, and we run into trouble exactly one unit from the center, we should realize we cannot go more than one unit left of the center either. (Whether or not the series converges on the boundary points $x = a \pm R$ needs to be looked at case-by-case, and quite possibly it converges at one but not both such endpoints.)

Furthermore, if we re-centered our series (14) for $1/(1+x^2)$ at, say, $x = 10$, then the nearest points where we lose analyticity in the complex plane are again at $\pm i$, which is a distance $|10-(\pm i)| = \sqrt{101}$ from our center, so our series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} b_k (x-10)^k$$

will converge for $|x - 10| < \sqrt{101}$, should we care to compute the coefficients $b_0, b_1, b_2, \ldots$.

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[5] For a simple example, think of $f(x) = \begin{cases} \frac{1}{2}x^2 & \text{for} \quad x \geq 0 \\ -\frac{1}{2}x^2 & \text{for} \quad x < 0 \end{cases} \implies f'(x) = |x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ -x & \text{for} \quad x < 0. \end{cases}$ That $f'(0) = 0$ requires some argument, after which we get $f'(0)$ exists but $f''(0)$, $f'''(0)$, etc., do not exist.

Homework 12-A

1. Compute the Taylor Series centered at $x = 0$ for

   (a) $\sin x^2$ and $\displaystyle\int_0^x \sin t^2\, dt$.

   (b) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$.

2. Consider $(1+x)^\alpha$. Its Taylor Series centered at $x = 0$ is called its Binomial Expansion.

   (a) Compute the Taylor Series centered at $x = 0$. (Write several terms and declare a pattern.)

   (b) Show that, if we take the first few binomial terms with the correct x and α, then we have

   $$\text{total energy} - \text{rest energy} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 \approx \frac{1}{2}mv^2,$$

   so the Einstein relativistic kinetic energy on the left is approximated by the Newtonian kinetic energy when $|v| << c$. (Note what happens to the total energy when $|v| \to c^-$.)