Lecture 13: Series Solutions near Singular Points

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Here we consider solutions to second-order ODE's using series when the coefficients are not necessarily analytic. A first-order analogy might be

$$y' + \frac{1}{x}y = 0 \implies \frac{y'}{y} = -\frac{1}{x} \implies \ln|y| = -\ln|x| \implies y = \frac{C}{x}.$$

So the coefficient of y was not analytic at zero,[1] and nor was the solution $y = C/x$. Note that the ODE above could have also been written $xy' + y = 0$, which looks less ill-behaved because all coefficients (x, 1) are analytic, but in fact there is a problem at $x = 0$, as we see in the solution. Indeed, it is important to isolate the highest-order derivative—or simply be sure its coefficient is 1—to detect such problems.

1 Poles and Singular Points of Functions

This section gives us a somewhat more sophisticated way of looking at singular behavior, to complement Zill's focused treatment. Suppose $g(x)$ is analytic at $x = a$, and $g(a) \neq 0$. Then

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots \\ a_0 &= g(a) \neq 0. \end{aligned}$$

Next consider the function

$$\begin{aligned} f(x) &= \frac{g(x)}{x-a} \\ &= \frac{1}{x-a}\sum_{n=0}^{\infty} a_k(x-a)^k \\ &= \sum_{n=0}^{\infty} a_k(x-a)^{k-1} \\ &= \sum_{n=-1}^{\infty} a_{k+1}x^k \\ &= \frac{a_0}{x-a} + a_1 + a_2(x-a) + a_3(x-a)^2 + a_4(x-a)^3 + \cdots . \end{aligned}$$

[1] Recall analytic means it can be written as its Taylor Series around that point, which implies all derivatives exist there, which implies the function and all its derivatives are continuous there. Recall

$$g'(a) \text{ exists } \implies g(x) \text{ continuous at } x = a.$$

So if a derivative exists, the function is continuous. Therefore if the second derivative exists, i.e., the derivative of the derivative exists, then the derivative itself is continuous. Similarly if g''' exists then g'' is continuous, and so on.

So a function which is undefined, or has a derivative of some order not existing at a point, can not be analytic at that point.

The final expression is not a Taylor/Power series *per se* anymore, since it starts with a negative power of $(x-a)$, and the first term is not $f(a)$ (which is undefined), the next term is not $f'(a)(x-a)$, etc. This new function $f(x)$ is said to have a *singularity* at $x=a$. In fact there are many types of singularities, and so to be more specific, $x=a$ is called a *pole of order 1* at $x=a$.

If instead we looked at $h(x)=g(x)/(x-a)^n$, where $n\in\{1,2,3,\dots\}$, we would call $x=a$ a pole of order n of the function $h(x)$. The higher we take n, the faster the function "blows up" as $x\to a$.

Another way to look a function with a pole at $x=a$ is to see how many factors of $x-a$ we should multiply the function with to result in an analytic function. For instance, in our example above we have (ignoring the removable discontinuity at $x=a$):

$$(x-a)f(x)=g(x),$$
$$(x-a)^n h(x)=g(x).$$

In fact, $(x-a)^2 f(x)=(x-a)g(x)$ is also analytic, but we look for the lowest power of $(x-a)$ which, multiplying $f(x)$, leaves us with an analytic function. Similarly, recalling $g(x)$ analytic at $x=a$ but $g(a)\neq 0$, we have

$$\begin{aligned}(x-a)^{n-1}h(x)&=\frac{g(x)}{x-a} && \text{is not analytic at} x=a,\\ (x-a)^n h(x)&=g(x) && \text{is analytic at } x=a,\\ (x-a)^{n+1}h(x)&=(x-a)g(x) && \text{is analytic at } x=a.\end{aligned}$$

So we can say the order of the pole at $x=a$ for a function $\eta(x)$, assuming it has a pole there, can be described as the smallest positive integer n so that $(x-a)^n\eta(x)$ is analytic at $x=a$.

Definition: We say $h(x)$ has a **pole of order** n at $x=a$ if and only if

1. $(x-a)^n h(x)$ is an analytic function (with a removable discontinuity at $x=a$), while[2]
2. $(x-a)^{n-1}h(x)$ is not analytic.

This is equivalent to the following.

Definition: We say $h(x)$ has a pole of order n at $x=a$ if and only if there exists $g(x)$ which is analytic at $x=a$, with $g(a)\neq 0$, and

$$h(x)=\frac{g(x)}{(x-a)^n}.$$

Note again that $h(x)$ can be written as a series of powers, but not as what we usually call a power series, since it won't actually be a Taylor Series. In fact, in Complex Analysis there is a topic called *Laurent Series,*[3] where we extend to include negative powers. Like the Taylor/Power series,

[2] Recall $x=a$ is a removable discontinuity of $f(x)$ means that

(1) $f(x)$ is discontinuous at $x=a$, but

(2) $\lim\limits_{x\to a} f(x)$ exists and is finite.

In such a case we can "redefine" $f(a)$ to be the limit in (1) above, and have a continuous function. For a simple example, consider $f(x)=(x^2-9)/(x-3)$, undefined at $x=3$, but otherwise the same as $\tilde{f}(x)=x+3$, but with a "hole" at $x=3$. Filling in the "hole" gives us a continuous function at $x=3$, and the discontinuity was "removed."

Another example is $f(x)=(\sin x)/x$, which is undefined at $x=0$, but has limit 1 there, so that discontinuity is removable.

[3] A perusal of Wikipedia gives this series as being named for Pierre Alphonse Laurent (1813–1854), French Mathematician though apparently discovered two years before attempted publication by the German mathematician Karl Weierstrass (1815–1897), whose contributions to analysis are legion. Laurent submitted the result for an academic prize but was past the deadline, and it was not actually published until after his death at age 41.

once its center is fixed the series is unique. The Laurent Series for $h(x)$ will be

$$h(x) = \frac{a_0}{(x-a)^n} + \frac{a_1}{(x-a)^{n-1}} + \cdots + \frac{a_{n-1}}{(x-a)} + a_n + a_{n+1}(x-a) + a_{n+2}(x-a)^2 + \cdots,$$

so one can read the order of the pole from the Laurent Series, and one could *define* the order of the pole as -1 times the first power that occurs without a nonzero coefficient. Note that the general Laurent Series is actually

$$\sum_{k=-\infty}^{\infty} C_k(x-a)^k = \cdots + \frac{C_{-2}}{(x-a)^2} + \frac{C_{-1}}{x-a} + C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots,$$

but for a pole of order n, only as many as n terms before the C_0 term would be nonzero.

All three definitions of the order of a pole have their uses, depending upon the context.

Note that there are functions which are "singular," or simply not analytic at $x = a$ but without having poles there. The functions $x^{2/3}$, $x^{-2/3}$, $\ln|x|$ are all nonanalytic at $x = 0$ (consider what their Taylor coefficients would be!), and all but the first are somehow "singular," i.e., "blowing up" at $x = 0$. There are also cases like

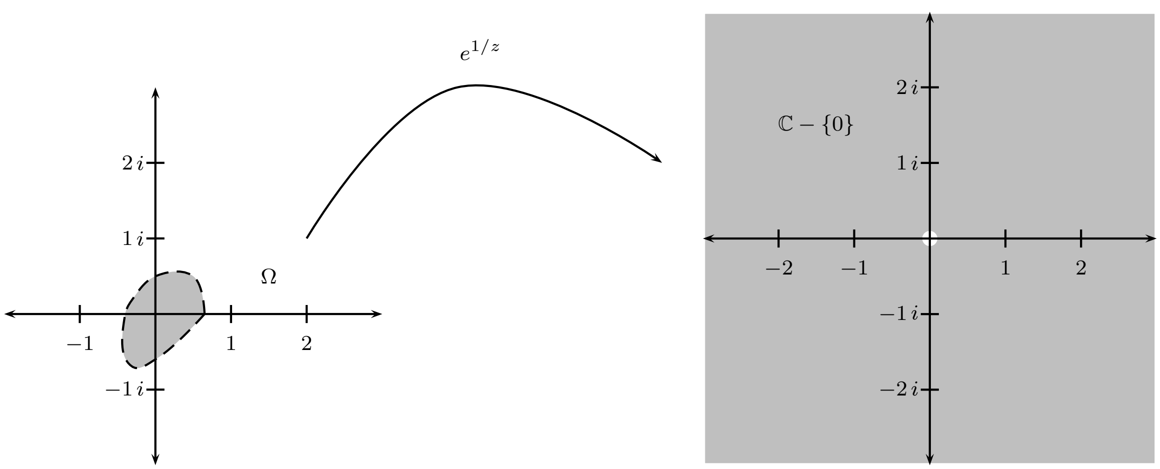
$$e^{1/x} = \sum_{k=0}^{\infty} \frac{x^{-k}}{k!} = 1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + \cdots.$$

The above expression is then the Laurent Series for $e^{1/x}$ written backwards. Some describe zero as a pole of infinite order for this function. In a complex variables setting it would be called an *essential singularity*.[4]

The function $(\sin x)/x^2$ actually has a pole of order 1 at zero, which is most easily seen by examining its Laurent Series:

$$\begin{aligned}\frac{\sin x}{x^2} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \frac{1}{x} - \frac{x}{3!} + \frac{x^3}{5!} - \frac{x^5}{7!} + \cdots.\end{aligned}$$

[4]One interesting thing about functions with essential singularities is that the image of any open set containing the singularity, under the function, will cover the whole complex plane, excepting possibly one point. In this case $e^{1/z} \neq 0$, but every other point in the complex plain is in the image of any open set Ω containing $z = 0$:



(Using Euler's formula, it is interesting to see how this function behaves as $z \to 0$ from different paths, such as along the imaginary axis, and then using real variable methods along the real axis. The other directions are more complicated.)

The fact that this series will represent $(\sin x)/x^2$ and is of the proper form for the Laurent Series tells us it must *be* the Laurent Series, and the pole is of order 1.

2 Frobenius' Theorem

The main topic of this lecture is a technique for solving second-order linear ODE's allowing for a certain level of singularity in the coefficients. The theorem which guarantees the existence of such solutions was given by Frobenius.[5] From Zill (p. 251), we have

Theorem: (Frobenius' Theorem) *If $x = x_0$ is a regular singular point of the differential equation*

$$y'' + P(x)y' + Q(x)y = 0, \tag{1}$$

then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}. \tag{2}$$

To understand the theorem, we need to know what is meant by *regular singular point* of (1).

Definition: $x = x_0$ *is called a* **regular singular point** *of* (1) *if and only if*

$$P(x) = \frac{p(x)}{x - x_0},$$
$$Q(x) = \frac{q(x)}{(x - x_0)^2},$$

where $p(x)$ and $q(x)$ are analytic functions at $x = x_0$.

Note that this means that $P(x)$ has at most a pole of order 1, and Q(x) a pole of at most order 2, at $x = x_0$. It is possible that they have lower-order poles, or one is analytic, but if both have no pole there then we can fall back on simpler, previous methods.[6] Of course, a singular point which is not "regular" by the above definition is an irregular singular point.

Referring back to (2), note that if $r \in \{0, 1, 2, 3, \ldots\}$, then we have an analytic solution. If $r \in \{-1, -2, -3, \ldots\}$ then we get a Laurent Series solution which is not a power series, i.e., we have a nonanalytic solution.

Also note that the theorem only guarantees we can find at least one such series solution (2). The method of finding another one is from Zill's Section 4.2, involving a technique called *reduction of order*, which we will pursue when it becomes relevant.

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[5] Ferdinand Georg Frobenius (1849–1917), German Mathematician who contributed to both differential equations and group theory.

[6] Zill's definition of regular singular point (p. 249) defines

$$p(x) = (x - x_0)P(x),$$
$$q(x) = (x - x_0)^2 Q(x)$$

being analytic. The definitions are equivalent, and have different ways of expressing the same nature of the singular points. Note that in neither definition is it stated that $p(x_0), q(x_0) \neq 0$.

3 Method of Frobenius

Here we look at Frobenius' actual method for finding a solution (2) to (1). In fact the method itself is straightforward, only requiring one extra observation compared to our previous series solutions. To summarize, assume y is in the form (2) and solves (2), and deduce the value of r, and from there (hopefully) a recursion relationship emerges from which we can write at least one of the two linearly independent solutions. Sometimes both emerge as series. Other times we need the reduction of order method to find the other.

Example 1 *Suppose we wish to solve the ODE*

$$2xy'' + y' + 4y = 0. \tag{3}$$

First we put this into a standard form (1):

$$y'' + \frac{1}{2x}y' + \frac{2}{x}y = 0.$$

We see that $x = 0$ is a regular singular point for our ODE. Using Frobenius' Theorem, we then assume a solution of the form $y = x^r \sum a_n x^n = \sum a_n x^{n+r}$, and plug this back into our ODE, either the standard form or the original form. We will use the original form (3) here. We get

$$2x\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Bringing the $2x$ into the first summation and combining some terms gives us

$$\sum_{n=0}^{\infty} a_n[2(n+r)(n+r-1) + (n+r)]x^{n+r-1} + 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

At this point we peel off the x^{r-1} $(n = 0)$ term from the first summation, begin that summation with the $n = 1$ term, but adjust our indices to start at $n = 0$ to match the second summation:

$$a_0[2(r)(r-1) + r]x^{r-1} + \sum_{n=0}^{\infty} \left\{ a_{n+1}[2(n+1+r)(n+r) + (n+1+r)] + 4a_n \right\} x^{n+r} = 0.$$

Note that the second summation gives a recursion relationship. Factoring inside that second summation, and setting the coefficient of x^{n+r} equal to zero, gives

$$\begin{aligned} & a_{n+1}(n+1+r)[2(n+r)+1] + 4a_n = 0 \\ \Longrightarrow \qquad & a_{n+1} = \frac{(-4)a_n}{(n+1+r)(2(n+r)+1)}. \end{aligned} \tag{4}$$

This is all well and good, but we need to find r. That is where the x^{r-1} term comes into play. It must be zero, so $a_0[2(r)(r-1) + r] = 0$, which means $a_0 = 0$ or $2r^2 - r = 0$. Now the ODE is linear and homogeneous, so we know that $y = 0$ is a solution, and by (4) we know that $a_0 = 0$ would just give us the trivial solution. Hence we instead conclude

$$2r^2 - r = 0. \tag{5}$$

Now (5) is called the **indicial equation**, *and gives us a lot of information about possible solutions. Note that we have two distinct roots to (5), since $r(2r-1) = 0 \iff r \in \{0, 1/2\}$. That these do not differ by an integer is important, and $r = 0$ and $r = 1/2$ will give us two linearly independent solutions. We will call them*

$$y_1 = x^0 \sum_{n=0}^{\infty} b_n x^n, \qquad y_2 = x^{1/2} \sum_{n=0}^{\infty} c_n x^n,$$

where b_n and c_n both satisfy the recursion relationship (4). We now find these in turn. For y_1, our recursion relation is (4) with $r = 0$:

$$b_{n+1} = \frac{(-4)b_n}{(n+1)(2n+1)}.$$

Beginning with b_0 (which is an arbitrary constant in this LHODE), we derive

$$\begin{aligned}
& & b_0 &= b_0 \\
(n=0): & & b_1 &= \frac{(-4)b_0}{1 \cdot 1} = -4b_0 \\
(n=1): & & b_2 &= \frac{(-4)b_1}{2 \cdot 3} = \frac{(-4)^2 b_0}{1 \cdot 2 \cdot 3} \\
(n=2): & & b_3 &= \frac{(-4)b_2}{3 \cdot 5} = \frac{(-4)^3 b_0}{(1 \cdot 2 \cdot 3) \cdot (3 \cdot 5)} \\
(n=3): & & b_4 &= \frac{(-4)b_3}{4 \cdot 7} = \frac{(-4)^4 b_0}{4! \cdot (1 \cdot 3 \cdot 5 \cdot 7)}.
\end{aligned}$$

At this point we claim to have established a pattern. Taking $b_0 = 1$, we can write

$$y_1 = \sum_{n=0}^{\infty} \frac{(-4)^n x^n}{n![(1)(3)(5) \cdots (2n-1)]},$$

which a ratio test would show is analytic on all of $\mathbb{R}$. Next we derive $y_2 = x^{1/2} \sum c_n x^n$. Since now $r = 1/2$ our recursion relation (4) becomes

$$c_{n+1} = \frac{(-4)c_n}{\left(n+1+\frac{1}{2}\right)(2n+2)} = \frac{(-4)^n c_n}{(2n+3)(n+1)}.$$

Beginning with c_0 we have

$$\begin{aligned}
& & c_0 &= c_0 \\
(n=0): & & c_1 &= \frac{(-4)c_0}{(3)(1)} \\
(n=1): & & c_2 &= \frac{(-4)c_1}{(5)(2)} = \frac{(-4)^2 c_0}{(1 \cdot 3 \cdot 5)(2!)} \\
(n=2): & & c_3 &= \frac{(-4)c_2}{(7)(3)} = \frac{(-4)^3 c_0}{(1 \cdot 3 \cdot 5 \cdot 7)(3!)}
\end{aligned}$$

Again taking our lead coefficient to be 1, we have

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-4)^n x^n}{n![(1)(3)(5) \cdots (2n+1)]}.$$

That y_1 and y_2 are linearly independent is obvious (neither is a constant multiple of the other), and we conclude $y = Ay_1 + By_2$ is our general solution. Now the summation in y_2 yields a function analytic on all of $x \in \mathbb{R}$, but the $x^{1/2}$ factor is undefined for $x < 0$, and not differentiable at $x = 0$. Therefore, the general solution is guaranteed to solve our ODE for $x > 0$. (However, if we assume $B = 0$ we have a solution for $x < 0$, from which we could find another with Reduction of Order methods.)

The above example only gives a very incomplete taste of the technicalities involved in the Frobenius Method. In fact there are cases, involving the ODE and the indicial equation. Rather than reproduce them all here, the student is directed to p. 255 of Zill, in the context of Section 6.2.