Lecture 14: The Laplace Transform

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Here we take a first look at the LaPlace[1]Transform. It should be pointed out that it is one of many transforms, a term which for our purposes means that it inputs whole functions and outputs functions as well. In fact, the input function is usually given as a function of an independent variable t (or x), and the output is a function of an entirely different independent variable s. The definition of the LaPlace Transform is as follows. Note that the notation $\equiv$ means equal "by definition," or "identically equal to."

$$\mathscr{L}\{f\} \equiv \int_0^\infty e^{-st} f(t)\,dt, \qquad s \geq 0. \tag{1}$$

In fact, to emphasize that f is a function of t, and its LaPlace Transform is a function of a new variable s, one usually writes instead

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

As a matter of notation, we will adopt the usual standard: anytime we input a function into the LaPlace Transform, we will use a lower-case letter for the function's name and the upper-case version of the same letter for its LaPlace Transform.

Note that the LaPlace Transform (1) is given by an improper integral, and there are often ranges for s for which it converges, and others for which it diverges.

Example 1 *Find the LaPlace transform of a function $f(t) = e^{\alpha t}$, where $\alpha \in \mathbb{R}$.*

<u>Solution</u>: We compute using the definition, but note that the integral only converges if $\alpha < s$:

$$\begin{aligned}\mathscr{L}\left\{e^{\alpha t}\right\} &= \int_0^\infty e^{-st} e^{\alpha t}\,dt = \int_0^\infty e^{(\alpha - s)t}\,dt = \lim_{\beta\to\infty} \int_0^\beta e^{(\alpha-s)t}\,dt = \lim_{\beta\to\infty} \frac{1}{\alpha - s} e^{(\alpha-s)t}\bigg|_0^\beta \\ &= 0 - \frac{1}{\alpha - s} = \frac{1}{s - \alpha}, \qquad s > \alpha.\end{aligned}$$

Note that we used the fact that $e^{(\alpha-s)\beta} \to 0$ as $\beta \to \infty$ if and only if $\alpha - s < 0$.

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[1] Pierre-Simon, Marquis de Laplace 1749–1827, French mathematician and astronomer. In Physics he is best known for the equilibrium heat equation: if $u(x, y, z)$ is the temperature at the point (x, y, z), in a uniform solid, then

$$\Delta u \equiv \nabla \cdot \nabla u \equiv u_{xx} + u_{yy} + u_{zz} = 0.$$

The differential operator on the left is called the *Laplacian.* (Of course the capital Greek Delta means other things in other contexts.)

LaPlace's equation also holds true if $u = u(x, y)$ in two dimensions, and so represents the equilibrium heat equation for a uniform "plate." Any function of $(x_1, x_2, \dots, x_n)$ which satisfies

$$\Delta u \equiv \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} u = 0$$

is called *harmonic.* Harmonic function theory is one of the most important theories in mathematics and physics. Two examples where harmonic function theory loom large are complex function theory and potential theory.

The LaPlace Transform was apparently discovered by Euler (1707–1783) previously. LaPlace used it extensively, including in his work in probability theory.

In fact this example above gives us two commonly used LaPlace transforms, the second one for the special case where $a = 0$:

$$\begin{array}{ccc} e^{at} & \xmapsto{\mathscr{L}} & \frac{1}{s+a}, \\ 1 & \xmapsto{\mathscr{L}} & \frac{1}{s}. \end{array}$$

As a general rule, the function $f(t)$ must grow, if at all, slower than some exponential function, i.e., we must have

$$|f(t)| \leq Me^{kt}$$

for some k, and then the integral in (1) will definitely converge, at least for every $s > k$.

One usually does not compute the LaPlace Transform "on the fly," while in the midst of working a problem for which the LaPlace Transform is a means to an end. Indeed, consider the effort for computing the LaPlace Transform for $\sin kt$. Beginning with the definition, we have

$$\mathscr{L}\{\sin kt\} = \int_0^\infty e^{-st} \sin kt \, dt,$$

for which we need integration by parts twice and some algebra to find the antiderivative:

$$\begin{aligned} \int e^{-st} \sin kt \, dt &= \int \underbrace{e^{-st}}_{u_1} \underbrace{\sin kt \, dt}_{dv_1} \\ &= \underbrace{e^{-st} \cdot \left(-\frac{1}{k} \cos kt \right)}_{u_1 v_1} - \int \underbrace{\left(-\frac{1}{k} \cos kt \right) (-se^{-st}) dt}_{v_1 \, du_1} \\ &= \frac{-e^{-st}}{k \cos kt} - \frac{s}{k} \int \underbrace{e^{-st}}_{u_2} \underbrace{\cos kt \, dt}_{dv_2} \\ &= \frac{-e^{-st}}{k \cos kt} - \frac{s}{k} \left[\underbrace{\left(e^{-st} \frac{1}{k} \sin kt \right)}_{u_2 v_2} - \int \underbrace{\left(\frac{1}{k} \sin kt \right) \left(-se^{-st} \right) \, dt}_{v_2 du_2} \right] \\ &= \frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt - \frac{s^2}{k^2} \int e^{-st} \sin kt \, dt. \end{aligned}$$

Adding the final integral to both sides we get

$$\begin{aligned} & \int e^{-st} \sin kt \, dt + \frac{s^2}{k^2} \int e^{-st} \sin kt \, dt = \frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt \\ \implies \quad & \left(1 + \frac{s^2}{k^2} \right) \int e^{-st} \sin kt \, dt = \frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt \\ \implies \quad & \left(\frac{k^2 + s^2}{k^2} \right) = \frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt \\ \implies \quad & \int e^{-st} \sin kt \, dt = \frac{k^2}{k^2 + s^2} \left[\frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt \right]. \end{aligned}$$

From the fact that $e^{-s\beta} \sin kt$, $e^{-s\beta} \sin kt \to 0$ as $\beta \to \infty$, we get

$$\int_0^\infty e^{-st} \sin kt \, dt = \lim_{\beta \to \infty} \frac{k^2}{k^2 + s^2} \left[\frac{-e^{-st}}{k \cos kt} - \frac{s}{k^2} e^{-st} \sin kt \right] \Bigg|_0^\beta = -\frac{k^2}{k^2 + s^2} \left[-\frac{1}{k} - 0 \right] = \frac{k}{k^2 + s^2}.$$

So after much effort, we get $\mathscr{L}\{\sin kt\} = \frac{k}{k^2+s^2}$.

Perhaps the example above helps to illustrate how it is perhaps best to construct, or refer to, a table of LaPlace Transforms when available, rather than computing these transforms each time one is needed.