Putnam Practice I: Polynomials

The problems on the Putnam Exam are a closely guarded secret. In fact the faculty who propose and select the problems are known as Committee X by the MAA. Thus, there is no way to know for sure which topics to study. However, there are certain ideas that make regular appearances. Over the next few weeks we will examine these topics with several practice problems.

Problems concerning polynomials are on almost every Putnam Exam. There are a few ideas that you should be familiar with so that you have some tools to use on these problems.

1. **The Zero-Coefficient Relationship** can be useful. Consider for an example a third degree polynomial $p(x)$ with zeros at r_1, r_2, and r_3. Suppose that the lead coefficient of this polynomial is one. Then we have

$$\begin{aligned} p(x) &= (x - r_1)(x - r_2)(x - r_3) \\ &= \left(x^2 - (r_1 + r_2)x + r_1 r_2\right)(x - r_3) \\ &= \left(x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_2 r_3 + r_1 r_3)x - r_1 r_2 r_3\right) \end{aligned}$$

   Notice that the coefficient of the second term is the opposite of the sum of all three root. The coefficient of the next term is the sum of the product of roots taken two at a time, and the last coefficient is the product of all three roots. A similar pattern holds for polynomials of degree n.

2. **The Fundamental Theorem of Algebra** says that if $p(x)$ is a polynomial of degree $n > 0$, in other words,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

   where the coefficients are real or complex, then $p(x)$ has at least one complex zero. In fact, we have the following.

   (a) $p(x)$ has precisely n zeros, given that a zero with multiplicity k is counted k times.

   (b) If the coefficients of $p(x)$ are all real, then whenever a complex number z is a zero of $p(x)$ then $\bar{z}$, its complex conjugate, is also a zero.

3. **Evaluating Polynomials to find Coefficients** can be useful for certain problems. Given a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

   we can see the following results.

   (a) $p(0) = a_0$. Evaluating at $x = 0$ gives you the constant term.

   (b) $p(1) = a_n + a_{n-1} + \cdots + a_0$. Evaluating the polynomial at $x = 1$ gives the sum of the coefficients.

   (c) $p(-1) = (-1)^n a_n + (-1)^{n-1} a_{n-1} + \cdots - a_1 + a_0$, the alternating sum of the coefficients.

4. **The Roots of Unity** satisfy an interesting polynomial. Take for example the zeros of $x^5 - 1$. We can see that $x = 1$ is a zero of this polynomial, and that means $x - 1$ is a factor. Dividing out gives

$$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1).$$

   Now we can see that $x = 1$ is not a zero of $x^4 + x^3 + x^2 + x + 1$, but the fundamental theorem of algebra says that $x^4 + x^3 + x^2 + x + 1$ has four complex zeros. These are the (fifth) roots of unity. They have the form $\omega = e^{\frac{2k\pi i}{5}}$ where $k = 1, 2, 3, 4$. They have the property that $\omega^5 = 1$. Thus, they are the remaining zeros of $x^5 - 1$, and also satisfy

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0.$$

The n-th roots of unity are similar. They have the form $\omega = e^{\frac{2k\pi i}{n}}$ for $k = 1, 2, \ldots n$. They are roots of the equations $x^n - 1 = 0$ and $x^{n-1} + x^{n-2} + \cdots + 1 = 0$.

5. **The Binomial Theorem** is also very helpful. Any textbook on College Algebra will have it, so make sure you check it out. It is recommended that you memorize the first few rows of Pascal's triangle so that you can perform the expansions expeditiously. For completeness, we give the general formula.

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

6. **The Remainder Theorem** tells us about what happens when we divide a polynomial $p(x)$ by a linear polynomial. Upon dividing by $x - c$, we can express $p(x)$ as

$$p(x) = q(x)(x - c) + 4,$$

where $q(x)$ is the quotient and r is the remainder. Substituting $x = c$ gives

$$p(c) = q(c)(c - c) + r.$$

Thus, the first term vanishes. We then have $p(c) = r$.

The following nine problems will provide some good practice with polynomials. They are grouped into three sections. Most people should be able to handle the warm-up problems without much difficulty. However, to get the maximum benefit from working on these exercises, you need to carefully write up your solution. This applies even to the warm-ups. Remember that to get credit for your work on the actual Putnam Exam, you will need to present your solution clearly and completely. Your solution must show all supporting reasoning in addition to your answer. Leaving out an important step will result in the loss of most of your credit. Thus, getting practice with this process will have benefits on test day. Writing up clear, complete solutions will also be beneficial on homework assignments for your various classes.

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Some Warm-Up Problems

1. Let $P(x)$ be a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?

2. Let $x_1 \neq x_2$ be such that $3x_1^2 - hx_1 = b$ and $3x_2^2 - hx_2 = b$. What is $x_1 + x_2$?

3. What is the remainder when $x^{51} + 51$ is divided by $x + 1$.

Some Harder Exercises

1. The parabola $y = ax^2 + bx + c$ has vertex (h, k). If we reflect about the line $y = k$, the result is the parabola $y_r = dx^2 + ex + f$. What is $a + b + c + d + e + f$?

2. Suppose that $P(x/3) = x^2 + x + 1$. What is the sum of all the values of x for which $P(3x) = 7$.

3. In $x^3 + px^2 + qx + r$, one zero is the sum of the other two. What is the relation between p, q and r?

4. If $x^2 + \dfrac{1}{x^2} = 14$ and $x > 0$, what is the value of $x^5 + \dfrac{1}{x^5}$

Real Putnam Problems

1. Prove that if $f(x)$ is a polynomial with integer coefficients, and there exists an integer k such that none of the integers $f(1), f(2), \ldots, f(k)$ is divisible by k, then $f(x)$ has no integral root.

2. Suppose that the function $f(x) = ax^2 + bx + c$, where a, b, c are real constants, satisfies the condition $|f(x)| \leq 1$ for $|x| \leq 1$. Prove that $|f'(x)| \leq 4$ for $|x| \leq 1$.