Putnam Exam Preparation and Strategy

1 Introduction

The William Lowell Putnam Mathematical Competition – referred to as the Putnam Exam by many – is the premier undergraduate inter-collegiate mathematics contest in North America. Last year over 4,200 students representing over 500 colleges and universities participated. Students who score highly on the Putnam receive cash prizes of up to $3,500 dollars. These students are often offered financial assistance to prestigious graduate programs.

The problems presented are challenging. To score highly, a participant must have not only technical competence in undergraduate mathematics, but also creativity. Many of the problems will use ideas typically presented in courses on calculus and linear algebra in novel ways. Further, at least some problems on every Putnam contain material from upper level courses like Modern Algebra, Real Analysis, or Topology.

2 Preparation

The best way to prepare for a tough exam like the Putnam is to work regularly on challenging problems. In your courses like Calculus and Linear Algebra, go through the exercise set and tackle the tougher problems in each section that your instructor covers, even if these are not assigned (that's good preparation for your in class exams too).

There are also some sources of problems that can be helpful.

MAA Minute Math is a blog that feature a problem from the American Mathematics Competition (AMC) every day. The AMC is an exam for high school students, so the content on the Putnam will be significantly more difficult. The reason I recommend this for preparation is so that solvers get used to tackling problems from a wide variety of areas.

A Primer for Mathematics Competitions by Zawaira and Hitchcock is a collection of problems and tools aimed at Math Olympiads. These competitions are also intended for high schoolers, but the problems are more challenging than those on the AMC.

Techniques of Problem Solving by Krantz deals with college level material. He breaks problems up into various categories and deals with strategies common for each group.

The William Lowell Putnam Mathematical Competition Problems and Solutions is a three volume collection of problems from the exam, along with their solutions, going back to 1985.

The American Mathematical Monthly publishes a collection of challenging problems each month. If you come up with a solution that is novel, you can submit it for publication.

3 Strategy for Test Day

Given the level of difficulty of the problems presented on the Putnam Exam, students who want to maximize their scores will need to have a plan of attack. Your instructor recommends following these steps during the exam.

1. In general, the problems are presented in order of increasing difficulty. In other words, problems A1 and A2 are likely to be the easiest problems you will see during the morning session. It pays to focus most of your attention on these first two problems, since you have the best chance at getting points from them.

2. Start by working on a problem that contains familiar ideas. Spend up to one half-hour thinking about concepts you know that might be relevant to the problem. Put these ideas down on scratch paper as they occur to you.

3. If you have nothing – no ideas about things that *might* help with a solution – after spending thirty minutes thinking about that one problem, then and only then should you move on to another.

4. If your solution is not presented clearly, you will get little or no credit. Start by getting your ideas down on scratch paper. To save time, and preserve a record of your thinking, do not erase any of your scratch work. Once you have some ideas down, begin working on putting them to use in your solution. This should still be on your scratch paper.

5. Once your are convinced that you have solved the problem, start using the folder to write your finished product.

6. Start each write-up by stating the solution to the problem. After that, give a careful justification of your solution. Make sure that you show all necessary steps while keeping out irrelevant material.

7. Even if you are not completely confident in your solution, write it up anyway. You might have the answer, or at least get some partial credit.

4 Last Year's Exam

A1 Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1,2,3,6\}, \{4,8\}, \{5,7\}$ shows that the largest k is *at least* 3.]

A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

A3 Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

A4 Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

A5 Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

A6 Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

B1 Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

B2 Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

B3 There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

B4 Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

B5 Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

B6 Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.