Chapter 10

Series of Constants

In this chapter we consider “infinite sums,” which we call series, (in both the singular and the plural) such as
\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots. \] (10.1)

The “sum” above begins with \( a_1 \), but we will often begin with a term \( a_0 \), or \( a_2 \), etc. It is not the beginning terms which determine if we can in fact compute such a sum, but rather it is the infinite “tail” of the series. This is reasonable because we can always, in principle, add as many terms together as we like, so long as there are finitely many of them. As with other calculus concepts, the tool which breaks the finite/infinite barrier is limit. Indeed, to make sense of a sum such as (10.1), we consider the \( N \)th partial sum,
\[ S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_{N-1} + a_N, \] (10.2)
and then look at the sequence of these partial sums \( S_1, S_2, S_3, \cdots \), i.e.,
\[ S_1 = a_1, \]
\[ S_2 = a_1 + a_2, \]
\[ S_3 = a_1 + a_2 + a_3, \]
\[ S_4 = a_1 + a_2 + a_3 + a_4, \]
and so on. To determine if (10.1) makes sense is then considered (by definition) to be equivalent to determining the behavior of the sequence \( \{S_N\}_{N=1}^{\infty} \). We say the series (10.1) converges to \( S \in \mathbb{R} \) if and only if \( S_N \to S \) as \( N \to \infty \).

In a few cases we will actually be able to compute a simple formula for \( S_N \), and thus be able to compute the series by taking \( N \to \infty \). However, in many cases we cannot find a compact formula for \( S_N \). In those cases we have to develop other methods for determining if the series converges at all, and if so, how to approximate the value of the series with as much precision as we require (short of exactness), by determining how large we require \( N \) to be so that we can approximate the full series by \( S_N \).

In this text we will take more steps than most other texts in developing the theory of series, since this topic is the source of much confusion for students. Indeed we devote this entire chapter to the topic of series of constant terms, leaving nonconstant terms for their own chapter. The concepts are intuitive—at times perhaps deceptively so—but require practice so that,
for example, one can recognize when and where to apply a particular test of convergence or divergence.

In the next chapter we will look at functions defined by series

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots. \] (10.3)

In fact most functions we have dealt with in this text can be written in the form above, at least on some open intervals, so such functions are very important theoretically. However there are very important functions which can be written in the form (10.3) but not by using the functions from our usual library (powers, logs, exponential, trigonometric and arctrigonometric functions). We deal extensively with functions of the form (10.3), also known as *power series*, in the next chapter.

One such function which can be represented by a series of form (10.3), as we will see, is the antiderivative \( F(x) \) of \( e^{x^2} \) whose graph passes through the origin, i.e., so that \( F(0) = 0 \). We will see how this can be given by the following, with \( a = 0 \) in (10.3):

\[
F(x) = \int_0^x e^{t^2} \, dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots.
\]

While this might appear complicated or intimidating, in fact it is a rather simple computation, though we must build some theory first. Of course, before dealing with series with variable terms, we must first develop a theory of series of constants, to which we devote the rest of this chapter.
10.1 Series and Partial Sums

As mentioned in the introduction to this chapter, the convergence of a series is defined as equivalent to the convergence of its partial sums. For convenience, we will define the \( N \)th partial sum to be the sum of all terms of the underlying sequence up to the term whose subscript is \( N \). Thus if the sequence is series is \( \sum_{n=k}^{\infty} a_n \), with underlying sequence \( \{a_n\}_{n=k}^{\infty} \), then

\[
S_N = \sum_{n=k}^{N} a_n = a_k + a_{k+1} + \cdots + a_N.
\]  

So for a series \( a_0 + a_1 + a_2 + \cdots \), the partial sum \( S_N = a_0 + a_1 + \cdots + a_N \) would actually have \( N + 1 \) terms, though we will still call it the \( N \)th partial sum. (Of course if \( N < k \) we do not define an \( N \)th partial sum.)

**Example 10.1.1** Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \), and find the first five partial sums.

**Solution**: We do this directly:

\[
S_1 = \sum_{n=1}^{1} \frac{(-1)^n}{n^2 + 1} = \frac{(-1)^1}{1^2 + 1} = \frac{-1}{2} = -0.5
\]

\[
S_2 = \sum_{n=1}^{2} \frac{(-1)^n}{n^2 + 1} = \frac{(-1)^1}{1^2 + 1} + \frac{(-1)^2}{2^2 + 1} = \frac{-1}{2} + \frac{1}{5} = \frac{-3}{10} = -0.3
\]

\[
S_3 = \sum_{n=1}^{3} \frac{(-1)^n}{n^2 + 1} = \frac{(-1)^1}{1^2 + 1} + \frac{(-1)^2}{2^2 + 1} + \frac{(-1)^3}{3^2 + 1} = \frac{-1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{2}{10} = 0.2
\]

\[
S_4 = \sum_{n=1}^{4} \frac{(-1)^n}{n^2 + 1} = \frac{(-1)^1}{1^2 + 1} + \frac{(-1)^2}{2^2 + 1} + \frac{(-1)^3}{3^2 + 1} + \frac{(-1)^4}{4^2 + 1} = \frac{-1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} \approx 0.2588235294
\]

\[
S_5 = \sum_{n=1}^{5} \frac{(-1)^n}{n^2 + 1} = S_4 + \frac{(-1)^5}{5^2 + 1} = \frac{44}{170} + \frac{-1}{26} \approx 0.220361991.
\]

Note we used the simple recursion relationship for partial sums of a series: given a series \( \sum_{n=k}^{\infty} a_n \), and \( N \geq k \) we have

\[
S_{N+1} = S_N + a_{N+1},
\]

that is,

\[
S_{N+1} = \sum_{n=k}^{N+1} a_n = a_k + a_{k+1} + \cdots + a_N + a_{N+1} = \sum_{n=k}^{N} a_n + a_{N+1} = S_N + a_{N+1}, \text{ q.e.d.}
\]

In a later section we will see that the series in the above example does in fact converge, though we can only approximate its exact value here by computing \( S_N \) for large values of \( N \).
10.1 SERIES AND PARTIAL SUMS

10.1.1 Telescoping Series

Telescoping series do occur on occasion, but the main reason they are included in most calculus textbooks is that their partial sums simplify in nice ways, leaving us able to compute their limits and thus the whole series. Indeed, the behavior of telescoping series is unusually “nice”—rivalled only by that of the much more important geometric series we will see later in this section—and therefore well-suited for early examples of the general notion of series.

The simplest type of telescoping series is one in which the terms added are themselves sums of two terms, constructed in such a way that there is cancellation such as the following:

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} [b_n - b_{n-1}]
\]

\[
= (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) + \cdots .
\] (10.6)

After a careful examination of the terms which appear in (10.6), it seems that all cancel except for \(-b_0\). However we must be more careful since there are infinitely many terms we are claiming we can cancel. The correct approach is to carefully examine the partial sums:

\[ S_1 = b_1 - b_0, \]
\[ S_2 = b_1 - b_0 + b_2 - b_1 = b_2 - b_0, \]
\[ S_3 = b_1 - b_0 + b_2 - b_1 + b_3 - b_2 = b_3 - b_0, \]
\[ S_4 = b_1 - b_0 + b_2 - b_1 + b_3 - b_2 + b_4 - b_3 = b_4 - b_0, \]

and so on, whereby we can conclude that, for this simplest type of example (10.6), we have

\[ S_n = b_n - b_0. \] (10.7)

Now such a series will therefore converge if and only if \(\{b_n\}_{n=1}^{\infty}\) converges. If \(b_n \to B \in \mathbb{R}\) as \(n \to \infty\), then by (10.7) we have \(S_n \to B - b_0\), whence \(\sum_{n=1}^{\infty} [b_n - b_{n-1}] = B - b_0\). More complicated telescoping series also occur, though the basic idea is that the partial sums can be written in such a way that all but a few terms found in the partial sums eventually cancel, and where we can compute the limits of those terms which do not.\(^1\) Rather than memorizing the sample telescoping forms (10.6) and (10.7), it is better to consider each example separately, writing out the terms of \(S_N\) for enough values of \(N\) that the pattern emerges.

**Example 10.1.2** Consider the series \(\sum_{n=1}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n} \right]\). Compute the form of each partial sum \(S_N\) (as a function of \(N\)), and the value of the series if it converges.

**Solution:** We will write out a few partial sums longhand, from which the pattern will emerge.

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\(^1\)It is interesting to visualize why the term *telescoping* is used to describe such a series. One of the Webster’s dictionaries defines the intransitive verb form of telescope as follows:

*to slide together, or into something else, in the manner of the tubes of a jointed telescope.*

For another example, a “telescoping antenna” comes to mind. Both can “collapse” to be much shorter than when fully extended. The reader should keep such images in mind as we consider so-called telescoping series.
Indeed, all but two terms will cancel in each of the following.

\[
S_1 = \left[1 - \frac{1}{2}\right] = \frac{1}{2} - 1,
\]

\[
S_2 = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{3} - \frac{1}{2}\right] = \frac{1}{3} - 1,
\]

\[
S_3 = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{3} - \frac{1}{2}\right] + \left[\frac{1}{4} - \frac{1}{3}\right] = \frac{1}{4} - 1,
\]

\[
S_4 = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{3} - \frac{1}{2}\right] + \left[\frac{1}{4} - \frac{1}{3}\right] + \left[\frac{1}{5} - \frac{1}{4}\right] = \frac{1}{5} - 1.
\]

From this we do indeed see a pattern in which

\[S_N = \frac{1}{N+1} - 1.\]

Taking \(N \to \infty\), we see \(S_N = \frac{1}{N+1} - 1 \to 0 - 1 = -1\), and so we conclude that the series converges to \(-1\), i.e.,

\[\sum_{n=1}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n}\right] = -1.\]

Sometimes we need to do a little more work to detect a telescoping series, and its formula for \(S_N\). Note that the general term of the added sequence terms, namely \(\frac{1}{n+1} - \frac{1}{n}\), in our series above looks like a partial fraction decomposition if the variable is \(n\). For that reason, when the general term can be written in a PFD, the series may in fact be telescoping. This is the case with the following example.

**Example 10.1.3** Consider the series \(\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}\). Compute a general formula for the \(N\)th partial sum \(S_N\), and compute its limit, if \(S_N\) converges, thereby computing the series.

**Solution:** Note first that there is no \(S_1\) here. That said, the technique which we will use for this is to first look at the partial fraction decomposition (PFD) for \(\frac{1}{n^2 - 1}\). Of course we need the denominator factored, giving us the form

\[
\frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}.
\]

Multiplying by \((n+1)(n-1)\) in the second equation then gives us

\[1 = A(n-1) + B(n+1).\]

Now we use the usual methods for computing the coefficients \(A\) and \(B\):

\[
\begin{align*}
\text{n = 1:} & \quad 1 = B(2) \quad \Rightarrow \quad B = \frac{1}{2} \\
\text{n = -1:} & \quad 1 = A(-2) \quad \Rightarrow \quad A = -\frac{1}{2}.
\end{align*}
\]

From this we can rewrite our series

\[
\sum_{n=2}^{\infty} \left[\frac{-1/2}{n+1} + \frac{1/2}{n-1}\right] = \sum_{n=2}^{\infty} \left[\frac{1}{2} \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right].
\]
There is no $S_1$, so we begin with $S_2$. (For space considerations we do not write out all terms at each line.)

$S_2 = \frac{1}{2} \left( -\frac{1}{3} + 1 \right) = \frac{1}{2} \left( \frac{-1}{3} + 1 \right)$,

$S_3 = \frac{1}{2} \left( -\frac{1}{3} + 1 \right) + \frac{1}{2} \left( -\frac{1}{4} + \frac{1}{2} \right) = \frac{1}{2} \left( \frac{-1}{3} + 1 + \frac{-1}{4} + \frac{1}{2} \right)$,

$S_4 = \frac{1}{2} \left( -\frac{1}{3} + 1 \right) + \frac{1}{2} \left( -\frac{1}{4} + \frac{1}{2} \right) + \frac{1}{2} \left( \frac{-1}{5} + \frac{1}{3} \right) = \frac{1}{2} \left( 1 + \frac{-1}{2} - \frac{1}{4} - \frac{1}{5} \right)$,

$S_5 = S_4 + \frac{1}{2} \left( -\frac{1}{6} + \frac{1}{4} \right) = \frac{1}{2} \left( 1 + \frac{-1}{2} - \frac{1}{5} - \frac{1}{6} \right)$,

$S_6 = S_5 + \frac{1}{2} \left( -\frac{1}{7} + \frac{1}{5} \right) = \frac{1}{2} \left( 1 + \frac{-1}{2} - \frac{1}{6} - \frac{1}{7} \right)$,

$S_7 = S_6 + \frac{1}{2} \left( -\frac{1}{8} + \frac{1}{6} \right) = \frac{1}{2} \left( 1 + \frac{-1}{2} - \frac{1}{7} - \frac{1}{8} \right)$

By this point a pattern has clearly emerged, and it can be written

$$S_N = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1} \right),$$

and so $S_N \rightarrow \frac{1}{2} \left[ 1 + \frac{1}{2} - 0 - 0 \right] = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$ as $N \rightarrow \infty$. We can thus conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \sum_{n=2}^{\infty} \left[ \frac{1}{2} \left( \frac{-1}{n+1} + \frac{1}{n-1} \right) \right] = \frac{3}{4}.$$
10.1.2 Geometric Series

The class of series considered here is arguably the most important we will encounter. Many important series analyses depend upon how a particular series compares to, or mimics the behavior of, an appropriately chosen geometric series. As with the telescoping series, the geometric series is one for which we can actually compute a general formula for $S_N$, from which we can tell if the series converges, and if so compute its sum.

What makes a series $\sum a_n$ geometric is that there exists a constant $r \in \mathbb{R} - \{0\}$ such that

$$(\forall n) \left[ \frac{a_{n+1}}{a_n} = r \right].$$

(10.8)

In other words, such a series can be defined recursively by $a_{n+1} = r \cdot a_n$. (Note that this is equivalent to $a_n = r \cdot a_{n-1}$, so long as $a_{n-1}$ is defined.) Put more colloquially, a geometric series is one in which we get the next term by multiplying the present term by the same constant each time. Examples of geometric series follow:

- $\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \quad (r = 1/2),$
- $\sum_{n=2}^{\infty} \frac{6}{5^n} = \frac{6}{25} + \frac{6}{125} + \frac{6}{625} + \cdots \quad (r = 1/5),$
- $\sum_{n=1}^{\infty} \frac{2(-1)^n}{3^n} = -\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \cdots \quad (r = -1/3),$
- $\sum_{n=1}^{\infty} \frac{1}{3^{2n}} = \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \frac{1}{6561} + \cdots \quad (r = 1/9).$

Note that this last series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{3^n}$, or even $\sum_{n=0}^{\infty} \left[ \frac{1}{9} \cdot \left( \frac{1}{3} \right)^n \right]$. In fact, unlike the telescoping series, every geometric series can be written in the same form, namely

$$\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots,$$

(10.9)

where

- $\alpha$ is the first term of the series, and
- $r$ is the constant ratio, $a_{n+1}/a_n$.  

(10.10)

In the examples above, the first terms are $\alpha = 1, 6/25, -2/3, 1/9$ respectively. Each of the series above can be rewritten in $\Sigma$-notation in the form (10.9), starting with $n = 0$. For instance, the third series above can be rewritten, using $\alpha = -2/3$ and $r = -1/3$, as

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{3^n} = -\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \cdots = \sum_{n=0}^{\infty} \frac{-2}{3} \left( \frac{-1}{3} \right)^n.$$

In fact, once we know a series is geometric (that is, that $a_{n+1} = r \cdot a_n$ for each $n$), all we need to do is to identify $\alpha$ and $r$, and we can write the series in the exact $\Sigma$-notation form (10.9).

With geometric series, it is understood that $"0^0"$ represents 1, even though technically this is only correct if $r > 0$. In each general setting in which we follow the convention that $r^0$ is defined to be 1 (regardless of the sign of $r$), we will remark on this point.
Example 10.1.5 Write the series \( 4 + \frac{2}{3} + \frac{1}{3} + \frac{1}{54} + \frac{1}{324} + \cdots \) in the form (10.9).

Solution: Though perhaps not immediately obvious, in fact each successive term is \( \frac{1}{6} \) times its immediate predecessor. The first term is 4. We translate these two facts as \( \alpha = 4 \) and \( r = \frac{1}{6} \), and so this series is the same as the series

\[
\sum_{n=0}^{\infty} 4 \cdot \left( \frac{1}{6} \right)^n.
\]

As with telescoping series, a geometric series allows for a simple formula for \( S_N \). To use the formula, however, we need to make two assumptions:

1. that the series is already written in the form
\[
\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots,
\]

and
2. that \( r \neq 1 \).

As we have seen, the first requirement is easy enough to accomplish: we need only identify \( \alpha \) (the first term in the geometric series) and \( r \). The second requirement is for technical reasons we will encounter momentarily. We do not lose much in assuming \( r \neq 1 \), since in the case \( r = 1 \) the series is simply \( \alpha + \alpha + \alpha + \cdots \), which is clearly a divergent series if \( \alpha \neq 0 \), and trivial if \( \alpha = 0 \).³ Now we state our theorem.

Theorem 10.1.1 For a geometric series
\[
\sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots,
\]
assuming \( r \neq 1 \), we have

\[
S_N = \frac{\alpha (1 - r^{N+1})}{1 - r}.
\]

Proof: The usual method of proof of (10.12) is to exploit the geometric nature of the series in the following way:

\[
\begin{align*}
S_N &= \alpha + \alpha r + \alpha r^2 + \cdots + \alpha r^N \\
r \cdot S_N &= \alpha r + \alpha r^2 + \alpha r^3 + \cdots + \alpha r^{N+1} \\
(1 - r)S_N &= (1 - r) \sum_{n=0}^{N} \alpha r^n \\
&= \alpha + 0 + 0 + 0 - \alpha r^{N+1}
\end{align*}
\]

In the first line we wrote the definition of \( S_N \). In the next line we multiplied that equation by \( r \). In the third line, the second line is subtracted from the first. In doing so, the terms \( \alpha r, \alpha r^2, \cdots, \alpha r^N \) cancel, leaving only \( \alpha - \alpha r^{N+1} \) on the right-hand side. This gives us

\[
(1 - r)S_N = \alpha (1 - r^{N+1}).
\]

Since we are assuming \( r \neq 1 \), we can divide by \( 1 - r \) and get (10.12), as desired.

To utilize (10.12), one needs to know \( \alpha \), \( r \) and \( N \). Note that \( N \) is not the number of terms, but the highest power of \( r \) which occurs. In fact there are \( N + 1 \) terms added to arrive at \( S_N \), since the first is \( \alpha r^0 \).

³We will not generally consider the case \( \alpha = 0 \) because it is trivial, and because we cannot identify a unique \( r \). Indeed, if \( \alpha = 0 \), then any geometric recursion \( a_{n+1} = r \cdot a_n \) is valid, but our original method of defining \( r \), namely (10.8) on page 706, is undefined if \( \alpha = 0 \).
**Example 10.1.6** Consider the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Find the sum of the first 9 terms.

**Solution:** What we are seeking here is $S_N = \frac{\alpha(1-r^{N+1})}{1-r}$, where $\alpha = 1$ and $r = \frac{1}{2}$. Thus

$$S_8 = \frac{1 \left[ 1 - \left( \frac{1}{2} \right)^9 \right]}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{512}}{\frac{1}{2}} = \frac{1}{2} \cdot 512 = \frac{512 - 1}{256} = \frac{511}{256} = 1.99609375$$

The formula (10.12) also works when $r < 0$.

**Example 10.1.7** Consider the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$. Find the sum of the first 9 terms.

**Solution:** Again we want $S_N$, but while $\alpha = 1$ as before, here we have $r = -1/2$.

$$S_8 = \frac{1 \left[ 1 - \left( -\frac{1}{2} \right)^9 \right]}{1 - (-\frac{1}{2})} = \frac{1 - \frac{1}{512}}{\frac{3}{2}} = \frac{1 + \frac{1}{512}}{\frac{3}{2}} \cdot \frac{512}{512} = \frac{512 + 1}{256 \cdot 3} = \frac{513}{256 \cdot 3} = \frac{171}{256} = 0.66796875$$

**Example 10.1.8** Suppose one deposits into an account (without interest) one penny ($0.01) on the first day of a month, then deposits two pennies ($0.02) the next day, four pennies the next, and so on, each day depositing twice what was deposited the day before. How much money is in the account after the first week (7 payments), second week, third week, and thirty-first day?

**Solution:** This is the same as asking for partial sums of the series $0.01 + 0.02 + 0.04 + 0.08 + \cdots$. This is a geometric series (10.9) with $\alpha = 0.01$ and $r = 2$. Here we have to be careful about $N$, since after the first day $N = 0$, after the second $N = 1$, etc. Now we compute the total deposit after

- 1 week, i.e., 7 days, we have $N = 6$ and
  $$S_6 = \frac{0.01 \left[ 1 - 2^7 \right]}{1 - 2} = \frac{0.01 \left[ 1 - 2^7 \right]}{-1} = 0.01(2^7 - 1) = 0.01(127) = 1.27.$$  

- 2 weeks, i.e., 14 days, we have $N = 13$ and (continuing the pattern above)
  $$S_{13} = \frac{0.01 \left[ 1 - 2^{14} \right]}{1 - 2} = 0.01(2^{14} - 1) = 0.01(16383) = 163.83.$$  

- 3 weeks, i.e., 21 days, we have $N = 20$ and
  $$S_{20} = \cdots = 0.01(2^{21} - 1) = 0.01(2,097,151) = 20,971.51$$

- 31 days, so we have $N = 30$, and
  $$S_{30} = \cdots = 0.01(2^{31} - 1) = 0.01(2,147,483,647) = 21,474,836.47.$$  

This latest example illustrates that, when $r > 1$, the function $N \mapsto S_N$ is essentially exponential. Indeed, as a function of $N$,

$$S_N = \frac{\alpha}{1 - r} \left[ 1 - r^{N+1} \right] = \frac{\alpha}{1 - r} - \frac{\alpha \cdot r^{N+1}}{1 - r} = \frac{\alpha}{1 - r} + \left[ \frac{\alpha}{r - 1} \right] \cdot r^N = A + Br^N,$$

where $A = \frac{\alpha}{r - 1}$ and $B = \frac{\alpha \cdot r}{r - 1}$. Thus as a function of $N$, $S_N$ is basically a vertical translation of an exponential growth $Br^N$, assuming again that $r > 1$. This partially explains why some use the term “geometric growth” when referring to exponential growth.

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4If $r \in (0, 1)$ we get a translation of exponential decay; if $r \in (-1, 0)$ we get a kind of “damped oscillation”; if $r = -1$ we get steady oscillation; and if $r < -1$ we get a growing oscillation. Details are left to the reader.
10.1. SERIES AND PARTIAL SUMS

10.1.3 Convergence/Divergence in Geometric Series

Now we look at necessary and sufficient conditions for a geometric series to converge. If a given geometric series does converge, we compute its sum. Our result is the following:

Theorem 10.1.2 For a geometric series \( \sum_{n=0}^{\infty} \alpha r^n = \alpha + \alpha r + \alpha r^2 + \alpha r^3 + \cdots \), where \( \alpha \neq 0 \),

1. the series converges if and only if \( |r| < 1 \), i.e., \( r \in (-1,1) \);
   equivalently, the series diverges if and only if \( |r| \geq 1 \), i.e., \( r \in (-\infty,-1] \cup [1,\infty) \).
2. if \( |r| < 1 \), then the series converges to \( \frac{\alpha}{1-r} \).

Restated, the geometric series converges to \( \frac{\alpha}{1-r} \) if \( |r| < 1 \), and diverges otherwise.

Proof: The proof requires some care, as the various cases contain their own technicalities.

- Case \( r = 1 \). In such a case, it is not difficult to see (we just count the terms!) that
  \[ S_N = \sum_{n=0}^{N} \alpha = (N+1)\alpha \to \infty \quad \text{as } N \to \infty. \]
  Thus \( r = 1 \) gives a divergent series.

- Case \( r = -1 \). In such a case, we have
  \[ \sum_{n=0}^{\infty} \alpha(-1)^n = \alpha - \alpha + \alpha - \alpha + \cdots, \]
  and so
  \[ S_N = \begin{cases} 
  \alpha, & \text{if } n \text{ is even}, \\
  0, & \text{if } n \text{ is odd}.
  \end{cases} \]
  In other words, \( \{S_N\}_{N=0}^{\infty} = \alpha,0,\alpha,0,\alpha,0,\cdots \), which is clearly a divergent sequence, i.e., the series itself is divergent (by definition).\(^5\)

- Case \( |r| > 1 \). Here we can use the formula for the partial sums:
  \[ S_N = \frac{\alpha (1 - r^{N+1})}{1-r}. \]
  Now there is only one term which is not a fixed constant, and so the convergence of this expression depends upon only the convergence that, \( r^{N+1} \)-term. Clearly if \( r > 1 \), this is an exponential growth, and diverges. For the general case \( |r| > 1 \), we get that\(^6\)
  \[ r^{N+1} \text{ converges } \Rightarrow \ |r|^{N+1} \text{ converges } \iff \ |r|^{N+1} \text{ converges.} \quad (10.13) \]
  But for \( |r| > 1 \), we have \( |r|^{N+1} \) diverges, so with the contrapositive of (10.13) we have
  \[ \begin{align*}
  |r| > 1 & \implies |r|^{N+1} \text{ diverges} \implies r^{N+1} \text{ diverges} \\
  \implies S_N = \frac{\alpha (1 - r^{N+1})}{1-r} \text{ diverges} & \iff \sum_{n=0}^{\infty} \alpha r^n \text{ diverges.}
  \end{align*} \]

\(^5\)Recall that the convergence of the series is defined by the convergence of the (sequence of) partial sums.

\(^6\)This follows from continuity of the function \( x \mapsto |x| \) giving us the “\( \implies \)” See Theorem 3.10.2, page 286.
• Case $|r| < 1$. Again we look at the variable part of the formula for $S_N$. It is enough to show that $|r| < 1 \implies r^{N+1}$ converges. One method is to use the sandwich theorem. In the argument below, note that $|r| < 1 \implies |r| \in (0,1) \implies r^{N+1} \to 0$. The relevant sandwich theorem application is then (as $N \to \infty$):

$$-|r|^{N+1} = -|r^{N+1}| \leq r^{N+1} \leq |r|^{N+1}$$

Thus $|r| < 1 \implies r^{N+1} \to 0$ as $N \to \infty$. We can conclude that

$$|r| < 1 \implies S_N = \frac{\alpha(1-r^{N+1})}{1-r} \to \frac{\alpha(1-0)}{1-r} = \frac{\alpha}{1-r} \text{ (as } N \to \infty).$$

This completes the proof.

The implication above is worth repeating in a summarized form:

$$|r| < 1 \implies \sum_{n=0}^{\infty} \alpha r^n = \frac{\alpha}{1-r}.$$

(10.14)

Also worth mentioning:

$$|r| \geq 1, \alpha \neq 0 \implies \sum_{n=0}^{\infty} \alpha r^n \text{ diverges.}$$

(10.15)

**Example 10.1.9** Here are some series computations using the theorem and (10.14).

• $\sum_{n=0}^{\infty} 2 \left(\frac{1}{3}\right)^n = \frac{2}{1-\frac{1}{3}} = \frac{3}{2} = 2 \cdot \frac{3}{2} = 3. \quad (\alpha = 2, r = \frac{1}{3})$

• $\sum_{n=0}^{\infty} 0.99^n = \frac{1}{1-0.99} = \frac{1}{0.01} = 100. \quad (\alpha = 1, r = 0.99)$

• $\sum_{n=0}^{\infty} 1.01^n \text{ diverges.} \quad (\alpha = 1, r = 1.01 \text{ so } |r| > 1, \text{ and the series diverges.})$

• $\sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{3}{2} = \frac{1}{6}. \quad (\text{First term is } \alpha = \frac{1}{3}, r = \frac{1}{3})$

• $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{\frac{1}{2}} = \frac{2}{3}. \quad (\alpha = 1, r = -\frac{1}{2})$

• $\sum_{n=1}^{\infty} e^{-n} = \frac{e}{1-\frac{1}{e}} = \frac{e}{\frac{e}{e}} = \frac{e^2}{e-1}. \quad (\alpha = e, r = \frac{1}{e})$

---

7Recall that for any $x \in \mathbb{R}$, we have $-|x| \leq x \leq |x|$. 
10.1. SERIES AND PARTIAL SUMS

\[ \sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} \frac{5}{9^n} = \frac{5/9}{1 - 1/3} = \frac{5/9}{8/9} = \frac{5}{8}, \quad (\alpha = \frac{5}{9}, r = \frac{1}{3}):\]

\[ \sum_{n=0}^{\infty} \frac{(-5)^n}{4^{2n+1}} = \frac{1}{4} - \frac{5}{4^3} + \frac{5^2}{4^5} - \frac{5^3}{4^7} + \cdots = \frac{1/16}{1 + 5/16} = \frac{4}{21}. \quad (\alpha = \frac{1}{4}, r = -\frac{5}{16}).\]

**Exercises**

1. Show that the following series can be written as a telescoping series, and discuss its convergence:

\[ \sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right). \]

2. Do the same with the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6}. \]

3. Do the same with the series

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+2)}. \]

4. Do the same with the series

\[ \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \frac{1}{n+3} \right]. \]

5. Find the \(N\)th partial sum of

\[ \sum_{n=0}^{\infty} 3^n, \]

and use it to determine if the series converges or diverges.

6. Do the same for the series

\[ \sum_{n=1}^{\infty} \frac{2}{3^n}. \]

7. For each, determine if the series converges or diverges, and if it converges, what is its sum (that it converges to).

(a) \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\)

(b) \(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\)

(c) \(100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots\)

(d) \(100 - 10 + 1 - \frac{1}{10} + \frac{1}{100} - \cdots\)

(e) \(\frac{1}{1000} + \frac{1}{100} + \frac{1}{10} + 1 + 10 + 100 + \cdots\)

8. For each of the following geometric series, find the first term and the ratio. Also determine if it converges or diverges, and if the former, find its sum. (For some of these, it might help to write out a few terms.)

(a) \(\sum_{n=2}^{\infty} \frac{2}{(-3)^n}\)

(b) \(\sum_{n=0}^{\infty} \left[ (-1)^n \left( \frac{4}{5} \right)^n \right]\)

(c) \(\sum_{n=1}^{\infty} \frac{9}{2^{2n}}\)

(d) \(\sum_{n=0}^{\infty} 5 \cdot \left( \frac{-4}{5} \right)^{2n+1}\)

(e) \(\sum_{n=0}^{\infty} \frac{3^n}{2^{3n-1}}\)

(f) \(\sum_{n=2}^{\infty} \frac{3^n}{(-2)^{n+1}}\)

(g) \(\sum_{n=0}^{\infty} \frac{3^n}{2^{2n+1}}\)

9. Give an alternative proof of the formula (10.12) for the partial sums of geometric series. For this new proof, begin with the formula for \(S_N\) as in the original proof (page 707), and then multiply by \((1 - r)\), noting how the right-hand side simplifies. (See also page 93.)
CHAPTER 10. SERIES OF CONSTANTS

Figure 10.1: Illustrations for showing that \( \sum \frac{1}{n} \) diverges while \( \sum \frac{1}{n^2} \) converges. We represent the terms of the series to be summed as areas of rectangles and show how they can be, respectively, underestimated or overestimated by areas under their respective curves. See the explanation following Example 10.2.1.

10.2 NTTFD and Integral Test

Because it is the exceptional case (e.g., geometric, telescoping) that we can actually find a compact formula for \( S_N \), we have to develop other tests for the convergence or divergence of series. There will be several such tests, and which particular test or tests are expeditious and conclusive will vary from series to series. We explore the first of those tests in this section. We start with two series that are similar, though one converges and the other diverges.

Example 10.2.1 The following are facts regarding two particular series.

- \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.
- \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

These are not obvious at all, but need to be proven. We do so by comparing the terms in the series—represented by areas of rectangles—to certain improper integrals, as in Figure 10.1.

For \( \sum_{n=1}^{\infty} \frac{1}{n} \), from the left-hand graph of Figure 10.1 we observe:

\[
S_1 = \frac{1}{1} \geq \int_{1}^{2} \frac{1}{x} \, dx
\]

\[
S_2 = \frac{1}{1} + \frac{1}{2} \geq \int_{1}^{3} \frac{1}{x} \, dx
\]

\[
S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \geq \int_{1}^{4} \frac{1}{x} \, dx
\]

\[
\vdots
\]

\[
S_N = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \geq \int_{1}^{N+1} \frac{1}{x} \, dx.
\]
Thus \( S_N \geq \int_{1}^{N+1} \frac{1}{x} \, dx = \ln(N+1) - \ln 1 = \ln(N+1) \to \infty \) as \( N \to \infty \).

We must conclude that \( S_N \to \infty \) as \( N \to \infty \), which implies that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (to infinity).

In fact for this example we have strict inequality “\( > \)" in each of the above, but it is enough that we have the non-strict “\( \geq \).”

For \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), instead we note from the right-hand graph of Figure 10.1 above that

\[
S_2 = \frac{1}{1} + \frac{1}{4} \leq 1 + \int_{1}^{2} \frac{1}{x^2} \, dx \\
S_3 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} \leq 1 + \int_{1}^{3} \frac{1}{x^2} \, dx \\
S_4 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \leq 1 + \int_{1}^{4} \frac{1}{x^2} \, dx \\
\vdots \\
S_N = \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{N^2} \leq 1 + \int_{1}^{N} \frac{1}{x^2} \, dx.
\]

Now clearly \( \{S_N\}_{N=1}^{\infty} \) is an increasing sequence, and \( \int_{1}^{N} \frac{1}{x^2} \, dx \) is obviously increasing with \( N \). Furthermore, for all \( N \)

\[
S_N \leq 1 + \int_{1}^{N} \frac{1}{x^2} \, dx \leq 1 + \int_{1}^{\infty} \frac{1}{x^2} \, dx. \tag{10.16}
\]

We can compute this improper integral as follows:

\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{\beta \to \infty} \int_{1}^{\beta} \frac{1}{x^2} \, dx = \lim_{\beta \to \infty} \left[ -\frac{1}{x} \right]_{1}^{\beta} = \lim_{\beta \to \infty} \left( -\frac{1}{\beta} + 1 \right) = 1. \tag{10.17}
\]

Putting computation (10.17) into (10.16), we get the upper bound

\[
S_N \leq 1 + 1 = 2.
\]

Thus \( \{S_N\}_{N=1}^{\infty} \) is a bounded, and obviously increasing sequence (since we are summing positive terms at each step), and therefore converges,\(^8\) so

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N \to \infty} S_N
\]

also converges, q.e.d. In fact, we even have an upper bound for the series:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2.
\]

However, we do not know from these computations its actual value. We only know that it does converge.\(^9\)

\(^8\) Recall that any bounded, increasing (or decreasing) sequence must converge to a limit. See Section 3.11.

\(^9\) Note that we used an improper integral to investigate the convergence or divergence of the series, but we did not compute the series’ exact value. We can get an estimate, meaning that we find a bound for it, but we cannot compute the exact value with these methods. There are methods for doing so, but they are indirect and beyond the scope of this text. In fact \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \), from Fourier Analysis, but it is a rather serendipitous computation that happens to give us that result, and not easily generalized. However, we can find how many terms must be added to be within a certain tolerance of the actual series’ value, as we will see in a later subsection.
10.2.1 First Divergence Test: NTTFD

When we compare the two series above, we see that both have terms that are shrinking to zero, but one series’ terms, \(1/n^2\) shrink much faster than the other’s, namely \(1/n\), just as \(1/x^2\) shrinks much faster than \(1/x\), hence the former’s improper integral on \([1, \infty)\) converges while the latter’s does not. In fact, according to our next theorem it is necessary for the terms to shrink if the series is to have any chance of converging. However, it is not enough that they shrink to zero; they have to shrink \textbf{fast enough} (assuming all terms are the same sign), or the series will diverge.

Thus it is necessary (even though not sufficient) that the terms of a series must shrink to zero if the series is to have any chance of converging. This we codify in a theorem.

**Theorem 10.2.1** Given a series \(\sum_{n=k}^{\infty} a_n\). If the series converges, then \(a_n \to 0\) as \(n \to \infty\):

\[
\sum_{n=k}^{\infty} a_n \text{ converges } \implies \left[ \lim_{n \to \infty} a_n = 0 \right]. \tag{10.18}
\]

**Proof:** Suppose \(\sum_{n=k}^{\infty} a_n\) converges, i.e., \(\sum_{n=k}^{\infty} a_n = L\) for some \(L \in \mathbb{R}\) (so in particular \(L\) is finite). Then by definition

\[S_N \to L \text{ as } N \to \infty.\]

Now recall \(S_n = a_n + S_{n-1}\), so that \(a_n = S_n - S_{n-1}\). Taking \(n \to \infty\) we get

\[a_n = S_n - S_{n-1} \to L - L = 0, \quad \text{q.e.d.}\]

This proof can seem “slick” to a novice mathematics student, but it is entirely correct. The proof notwithstanding, it should be intuitively clear that, if we are going to “add up” infinitely many terms, and have the sums approach a finite number, then the terms we are adding are going to have to shrink to zero, at least in the limit. The proof uses the fact that \(S_n \to L \implies S_{n-1} \to L\), the latter limit occurring “one step behind” the former, but occurring nonetheless since \(n \to \infty \implies n-1 \to \infty \implies S_{n-1} \to L\).

Note that it was important that \(L\) be finite in the limit computation above. For instance, if \(S_n \to \infty\), we would have \(a_n = S_n - S_{n-1}\) giving \(\infty - \infty\)-form (which is indeterminate) as \(n \to \infty\).

Again, the intuition behind the theorem is that, in order to be able to add infinitely many terms—one at a time in the sense that we compute \(S_k, S_{k+1}, S_{k+2}, \text{etc.}\), and look for a trend in these sums towards \(L\)—the terms that we add, i.e., \(a_k, a_{k+1}, a_{k+2}, \text{etc.}\), have to shrink eventually if the partial sums are to get closer and closer to (“approach”) a finite number \(L\).

In fact, the form of the theorem which we use is the contrapositive. Recall the logical equivalence \(P \to Q \iff (\sim Q) \to (\sim P)\).\(^1\) In this case, \(P\) is the statement that the series converges (to a finite number \(L\)), while \(Q\) is the statement that \(a_n \to 0\). The contrapositive for of Theorem 10.2.1 is our main result in this section, and we dub that result \textit{nth term test for divergence}, or NTTFD:

\(^\text{10}\)To be sure, here \(P \to Q\) is read, “\(P\) implies \(Q\).” The symbol \(\sim\) is still the “not,” or logical negation, operator. The symbol “\(\iff\)” stands in for logical equivalence. Recall \(P \to Q\) and \((\sim Q) \to (\sim P)\) are \textit{contrapositives} of each other, and are logically equivalent.
Theorem 10.2.2 (NTTFD) If it is not the case that \( a_n \to 0 \), then \( \sum_{n=k}^\infty a_n \) diverges. Put symbolically,

\[
a_n \not\to 0 \implies \sum_{n=k}^\infty a_n \text{ diverges.} \quad (10.19)
\]

**Proof:** It is enough to say that this is the contrapositive of Theorem 10.2.1, and therefore also true. One way to write this symbolically is the following.

\[
\sum_{n=k}^\infty a_n \text{ converges } \iff (a_n \to 0) \iff \left[ \sim (a_n \to 0) \right] \longrightarrow \sum_{n=k}^\infty a_n \text{ diverges.}
\]

The statement on the right must be a tautology (always true), since it is equivalent to the statement on the left, which—being the statement of Theorem 10.2.1—is itself a tautology. The statement on the right being a tautology means that it can stand alone as a tautology, written as in (10.19), q.e.d.

This theorem is undoubtedly one of the most misunderstood and misapplied results in all of Calculus I and II. It is as important to understand what it does not say, as it is to understand what it says. The theorem says that if the terms of a series do not shrink to zero, then the series must diverge.

But it is not as comprehensive as one might think. After all, if the terms of a series do shrink to zero, the theorem is silent! (Therein lies the unfortunately very common mistake made by calculus students.) To emphasize this we look at the following examples.

**Example 10.2.2** Discuss what Theorem 10.2.2 has to say about the series

(a) \( \sum_{n=1}^\infty \frac{n}{2n+1} \);

(b) \( \sum_{n=1}^\infty \frac{1}{n+1} \);

(c) \( \sum_{n=1}^\infty \cos \frac{1}{n} \);

(d) \( \sum_{n=1}^\infty \sin \frac{1}{n} \);

(e) \( \sum_{n=1}^\infty \sin \frac{1}{n^2} \).

**Solution:**

(a) \( \frac{n}{2n+1} \longrightarrow \frac{1}{2} \neq 0 \implies \sum_{n=1}^\infty \frac{n}{2n+1} \text{ diverges.} \)

(b) \( \frac{1}{n+1} \longrightarrow 0. \text{ The NTTFD is inconclusive.} \)

(c) \( \cos \frac{1}{n} \longrightarrow \cos 0 = 1 \neq 0 \implies \sum_{n=1}^\infty \cos \frac{1}{n} \text{ diverges.} \)

(d) \( \sin \frac{1}{n} \longrightarrow \sin 0 = 0. \text{ The NTTFD is inconclusive.} \)
(e) \( \sin \frac{1}{n^2} \rightarrow \sin 0 = 0 \). The NTTFD is inconclusive.

Looking closely at the symbolic statement of NTTFD given in (10.19), we see that there is never an implication of convergence. Indeed, the test either concludes divergence, or is inconclusive. This is a very quick but incomplete test, which can only detect divergence in certain (still common) circumstances, namely that \( a_n \neq 0 \).

Indeed, in (a) and (c) above, NTTFD gave us divergence. However, it said nothing in (b), (d) and (e), as the “if” part of the theorem was not true. In fact, of these three in which NTTFD is silent—(b), (d) and (e)—it turns out that (b) and (d) are divergent, while (e) is convergent. The methods to see this are introduced in later sections. An example from the exercises in the last section gives a case where we can in fact show that it is possible that \( a_n \rightarrow 0 \), but the series diverges:

**Example 10.2.3** Consider the series \( \sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right) \). Determine if it converges or diverges.

**Solution:** First we note \( \ln \left( \frac{n}{n+1} \right) \rightarrow \ln 1 = 0 \) as \( n \rightarrow \infty \). Thus NTTFD is inconclusive.\(^{11}\)

Looking closer at this series, we should eventually notice that it is telescoping. This becomes clear if we rewrite it:

\[
\sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right) = \sum_{n=1}^{\infty} \left[ \ln(n) - \ln(n+1) \right] = [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + [\ln 3 - \ln 4] + \cdots .
\]

It is not difficult to see that \( S_N = \ln 1 - \ln N = -\ln N \), and thus

\[
S_N = -\ln N \rightarrow -\infty \text{ as } N \rightarrow \infty.
\]

Since the partial sums diverge, by definition so does the series.

### 10.2.2 Integral Test Proper (IT)

Of course there are series which converge. However the NTTFD is never powerful enough to ever prove it (but can sometimes detect divergence in a series). Proving actual convergence requires other tests. The integral test is one such test:

**Theorem 10.2.3** Suppose we have a series \( \sum_{n=k}^{\infty} a_n \) such that

1. \( a_n = f(n) \) for each \( n \geq k \), where
2. \( f(x) \geq 0 \) is continuous and nonincreasing on \([k, \infty)\).

Then \( \sum_{n=k}^{\infty} a_n \) and \( \int_{k}^{\infty} f(x) \, dx \) both converge or both diverge. In other words,

\[
\sum_{n=k}^{\infty} a_n \text{ converges } \iff \int_{k}^{\infty} f(x) \, dx \text{ converges.}
\]

Equivalently,

\[
\sum_{n=k}^{\infty} a_n \text{ diverges } \iff \int_{k}^{\infty} f(x) \, dx \text{ diverges.}
\]

\(^{11}\)Some textbooks would write that the test “fails.” That seems a bit strong. It is merely inconclusive, so we need to look deeper at the particular series and perhaps employ some other test which will be conclusive.
10.2. NTTFD AND INTEGRAL TEST

**Proof:** We will not write the whole proof here, but just mention that it follows the same kind of reasoning we used to show \( \sum \frac{1}{n} \) diverges while \( \sum \frac{1}{n^2} \) converges. It is not too difficult to see—by drawings similar to those early examples—that

(a) If \( \int_k^\infty f(x) \, dx \) diverges, then it diverges to \( \infty \), and so as \( N \to \infty \) we have
\[
S_N \geq \int_k^{N+1} f(x) \, dx \to \infty,
\]
so \( \sum_{n=k}^\infty a_n \) diverges to infinity.

(b) If \( \int_k^\infty f(x) \, dx \) converges and is thus finite, then
\[
S_N \leq a_k + \int_k^N f(x) \, dx \leq a_k + \int_k^\infty f(x) \, dx < \infty,
\]
so \( S_N \) is a bounded, clearly nondecreasing sequence, so it must converge, implying (by definition) that \( \sum_{n=k}^\infty a_n \) converges.

**Example 10.2.4** Consider any series \( \sum_{n=1}^\infty \frac{1}{n^p} \), where \( p > 0 \). Then \( f(x) = \frac{1}{x^p} \) is clearly a decreasing function on \([1, \infty)\).\(^{12}\) Now
\[
\lim_{\beta \to \infty} \int_1^\beta x^{-p} \, dx = \left\{ \begin{array}{ll}
\lim_{\beta \to \infty} \frac{x^{1-p} \beta^{1-p}}{1-p} & = \frac{-1}{1-p} \text{ if } p > 1 \quad (1-p < 0) \\
\lim_{\beta \to \infty} \ln x^{1-p} & = \infty \quad \text{if } p = 1 \\
\lim_{\beta \to \infty} x^{1-p} \beta^{1-p} & = \infty \quad \text{if } p < 1 \quad (1-p > 0)
\end{array} \right.
\]
from which we get that \( \int_1^\infty \frac{1}{x^p} \, dx \) converges if and only if \( p > 1 \), and diverges if and only if \( p \leq 1 \). This gives us the following as well:

**Theorem 10.2.4** \( \sum_{n=1}^\infty \frac{1}{n^p} \) converges if and only if \( p > 1 \) (and diverges if and only if \( p \leq 1 \)).

Series of the form \( \sum_{n=1}^\infty \frac{1}{n^p} \) are called \( p \)-series. They are nearly as important as geometric series within the general theory of series. If one forgets about the two cases, the integral test, Theorem 10.2.3, page 716 makes deriving this last theorem fairly straightforward.

**Example 10.2.5** Determine if the series \( \sum_{n=2}^\infty \frac{1}{n \ln n} \) converges.

**Solution:** Clearly this is a series of positive terms, which decrease monotonically. We can look therefore at \( \int_2^\infty \frac{1}{x \ln x} \, dx \), since \( f(x) = 1/(x \ln x) \) is also positive and decreasing on \([2, \infty)\). Now
\[
\lim_{\beta \to \infty} \int_2^\beta \frac{1}{x \ln x} \, dx = \lim_{\beta \to \infty} \ln |\ln x|^\beta = \lim_{\beta \to \infty} [\ln(\ln \beta) - \ln(\ln 2)] = \infty,
\]
\(^{12}\)We can note that the denominator is increasing and positive, and so intuitively these fractions \( 1/x^p \) must shrink. On the other hand, we can also compute \( f'(x) = \frac{d}{dx} x^{-p} = -p \cdot x^{-p-1} = -p/(x^{p+1}) < 0 \), so the fact that \( f' < 0 \) on \([1, \infty)\) also shows that \( f(x) \) is decreasing there.
we conclude that \( \int_2^\infty \frac{1}{x \ln x} \, dx \) diverges to infinity, and therefore so does the original series, by the Integral Test.

We will have other tests in subsequent sections. However, for series which satisfy the hypotheses of the Integral Test, it is the most sensitive, and can determine convergence or divergence for many series for which other tests are inconclusive. (It already has shown convergence or divergence for series for which the NTTFD is silent.) It is also interesting to note that the Integral Test, or its general reasoning buried in the proof, can determine which geometric series converge and which diverge, assuming \( \alpha > 0, r > 0, \) and \( r \neq 1 \) (or the series obviously diverges).

\[
\lim_{\beta \to \infty} \int_0^\beta \alpha r^x \, dx = \lim_{\beta \to \infty} \frac{\alpha r^\beta}{\ln r} \bigg|_0^\beta \to \begin{cases} 
\infty & \text{if } r > 1 \\
\frac{\alpha}{\ln r} & \text{if } r \in (0, 1).
\end{cases}
\]

Note that for \( r \in (0, 1) \) we have \( \ln r < 0 \) so the integral is positive. So the Integral Test concludes that \( \sum \alpha r^n \) converges if \( r \in (0, 1) \) and diverges if \( r \geq 1 \). (If \( \alpha < 0 \) we just look at \( \sum \alpha r^n = -\sum (-\alpha)r^n \), and if \( \alpha = 0 \) the series obviously converges.) Unfortunately the Integral Test can not be used here for \( r < 0 \), but fortunately we have a formula (10.12), page 707 for \( S_N \) anyhow, from which we can judge convergence based upon what occurs if \( N \to \infty \).

**Example 10.2.6** Consider \( \sum_{n=0}^\infty \frac{1}{n^2 + 1} \), and discuss if it converges or diverges.

**Solution:** First note that \( a_n = f(n) > 0 \), where \( f(x) = \frac{1}{x^2 + 1} \) is a decreasing function on \([0, \infty)\). Again this is intuitive, but a quick check shows \( f' < 0 \) on \((0, \infty)\), which is enough to show \( f \) is decreasing on \([0, \infty)\).\(^{13}\)

Next we check the relevant improper integral.

\[
\int_0^\infty \frac{1}{x^2 + 1} \, dx = \lim_{\beta \to \infty} \int_0^\beta \frac{1}{x^2 + 1} \, dx = \lim_{\beta \to \infty} \left( \tan^{-1} \beta - \tan^{-1} 0 \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.
\]

Since the improper integral converges, so does the series \( \sum_{n=0}^\infty \frac{1}{n^2 + 1} \) converge.

One interesting question is how many terms of the series do we need to add in order to get a good approximation of the full series? We clearly can not add infinitely many, and this series is not telescoping and not geometric, so finding a simple formula for \( S_N \) for which we can let \( N \to \infty \) seems unlikely. So instead we look back to the integral that proves the series converges. While there seems to be a general formula, it is probably best practice (unless one does such problems repeatedly) to re-draw the situation when needed. Nonetheless, it is interesting to see the general rule:

**Theorem 10.2.5** Suppose \( \sum a_n \) satisfies the hypotheses of the Integral Test, in the sense that \( a_n = f(n) \) for some \( f(x) \) defined, nonnegative and nonincreasing on some interval \([k, \infty)\), and \( \int_k^\infty f(x) \, dx < \infty \). Then the series converges to some value \( \sum a_n = S \), and for \( N > k \) we have

\[
|S - S_N| = S - S_N \leq \int_N^\infty f(x) \, dx.
\]  

\(^{13}\)In fact, we really only need \( f(x) \) to be decreasing on some set \([k, \infty)\), since we can always sum the finitely many terms that occur before \( n = k \). It is the infinite “tail-end” sum of the series that determines convergence, i.e., we need to compute if \( \sum_{n=k}^\infty a_n \) converges. We have the liberty to ignore what happens for any finite number of terms at the “front” of the series, assuming they are defined.
10.2. NTTFD AND INTEGRAL TEST

\[ S - S_N = \sum_{n=N+1}^{\infty} a_n \leq \int_{N}^{\infty} f(x) \, dx \]

**Figure 10.2:** Here we look at the error \( S - S_N \) if we sum only the first terms up to and including \( a_N \). \( S_N \) is represented by the areas of the colored rectangles; the remaining “tail-end” of the series, i.e., the error \( S - S_N \) from missing the remaining terms, is represented by the remaining, unshaded rectangles, which are safely under the curve \( y = f(x) \). (For simplicity of illustration we show the case \( N = 4 \), and the series starts at \( n = 1 \).)

For our previous example, if we wished to know \( \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} \) to within 0.10, we can find \( N \) so that \( \int_{N}^{\infty} \frac{1}{x^2 + 1} \, dx \leq 0.10 \), and that will guarantee \( S - S_N \leq 0.10 \) as well. Now for any given \( N \), we have

\[
\int_{N}^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{\beta \to \infty} \int_{N}^{\beta} \frac{1}{x^2 + 1} \, dx = \lim_{\beta \to \infty} (\tan^{-1} \beta - \tan^{-1} N) = \frac{\pi}{2} - \tan^{-1} N.
\]

For \( \pi/2 - \tan^{-1} N \leq 0.01 \), we add \( \tan^{-1} N \) to both sides and see we need

\[ \tan^{-1} N \geq \frac{\pi}{2} - 0.10 \implies N \geq \tan \left( \frac{\pi}{2} - 0.10 \right) \approx 9.966644419, \]

so we take \( N \geq 10 \) for such accuracy.

If instead we wanted accuracy to within 0.001, we have a similar calculation yielding

\[ \tan^{-1} N \geq \frac{\pi}{2} - 0.001 \implies N \geq \tan \left( \frac{\pi}{2} - 0.001 \right) \approx 999.99962177, \]

so we would need to take \( N \geq 1000 \), and thus compute

\[
\sum_{n=0}^{1000} \frac{1}{n^2 + 1}
\]

to assure that we achieve such accuracy in using \( S_N \) to approximate the full series \( S \). Note that it quickly becomes more suitable for numerical (specifically, electronic) computational devices.\(^\text{14}\)

In a later section, we will note that the convergence of \( \sum \frac{1}{n^2} \) guarantees the convergence of \( \sum \frac{1}{n^2 + 1} \), because

\[ 0 \leq \frac{1}{n^2 + 1} \leq \frac{1}{n^2}. \]

\(^{14}\)In computing the inequality, we used that \( \tan x \) is an increasing function on \( (-\pi/2, \pi/2) \), so for all \( x_1, x_2 \in (-\pi/2, \pi/2) \), we have \( x_1 < x_2 \implies \tan x_1 < \tan x_2 \).
and since $1/n^2$ is summable, i.e., $\sum \frac{1}{n^2}$ converges ($p$-series, $p > 1$), the series of smaller terms must also. That is called the Direct Comparison Test, though there are still others.\footnote{As is often the case, we use “$\leq$” when we in fact have “$<$.” This is partly to show that “$\leq$” is sufficient here, though to be more precise we should use “$<$.”}

**Exercises**

1. Use the NTTFD to determine which of the following series must diverge (based only upon that test). If NTTFD is inconclusive, so state.

   (a) $\sum_{n=1}^{\infty} e^{1/n}$
   (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$
   (c) $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

2. Use the integral test to determine convergence or divergence of the following series.

   (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
   (b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
   (c) $\sum_{n=2}^{\infty} \frac{1}{n(n \ln n)^2}$
   (d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$
   (e) $\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$
   (f) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

3. It is known from other fields that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \approx 1.644 934 067$.

   (a) Find $S_{10}$, and estimate how accurate that is using the integral test estimate (10.20).
   (b) How large would $N$ need to be to ensure that $S_N$ is within 0.0001 of the full series?

4. The integral test requires that $f(x)$ be eventually monotonically decreasing. Here we give an example where the series and improper integral behave very differently, to show that we can not relax our hypotheses on $f$.

   (a) Show that $\int_{1}^{\infty} \sin^2(\pi x) \, dx$ diverges (to infinity).
   (b) Show that $\sum_{n=1}^{\infty} \sin^2(\pi \cdot n)$ converges.
   (c) Graph $y = \sin^2(\pi x)$, for $x \geq 0$. Explain why this is not a proper function to choose when determining the convergence of $\sum \sin(n\pi)$ with the integral test.
10.3 Comparison Tests

Before proceeding to the topic of this section, we briefly review what we have developed so far regarding series.

To say that $\sum a_n$ converges to some number $S \in \mathbb{R}$ is, by definition, to say that $S_N \to S$, where $S_N$ is the $N$th partial sum of the series. There were two—namely the telescoping and geometric series—for which we were able to compute $S_N$ and therefore its limit as $N \to \infty$, and therefore $\sum a_n$.

Next we noted that we could quickly detect divergence in some cases with the NTTFD, as $a_n \not \to 0 \implies \sum a_n$ diverges. Of course $a_n \to 0 \not \Rightarrow \sum a_n$ converges.

Then we realized that there are many series which we can prove converge, as in cases where $\{S_N\}$ is a bounded and increasing sequence, but we could not directly calculate the value of the full series. We proved these converged using integral tests, where we compared them to $\int_1^\infty f(x) \, dx$, where $a_n = f(n)$ for an appropriate function $f$. The series converged if and only if the improper integral did, but we were careful that, (1) $f(x)$ was eventually nonincreasing on some interval of the form $[k, \infty)$, and (2) we did not make the mistake of claiming that the series and the integral converged to the same number, but only that their behaviors were similar enough that convergence (or divergence) of one meant the same for the other.

After these developments, we made special note of two types of series for which we know immediately whether or not they converge (recall “iff” is short for “if and only if”):

1. geometric series: $\sum_{n=0}^\infty \alpha r^n$ converges iff $r \in (-1, 1)$, and diverges otherwise (here $\alpha \neq 0$);

2. $p$-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges iff $p > 1$, and diverges otherwise (i.e., for $p \leq 1$).

For the $p$-series, we proved this using the Integral Test, and gave some special mention to the borderline case, $p = 1$, mentioning that the harmonic series $\sum \frac{1}{n}$ diverges.

These two series types above will be very useful for both of our comparison tests developed in this section, especially the $p$-series. However, before we look at these we make another observation:

**Theorem 10.3.1** $\sum_{n=1}^\infty a_n$ converges $\iff (\forall M \in \{1, 2, 3, \cdots\}) \left[ \sum_{n=M}^\infty a_n \text{ converges} \right]$.

This is just the statement that the first $M - 1$ terms are not what determine convergence or divergence, no matter how large or small $M$ happens to be. It is the series’ tail end, containing infinitely many terms to sum, which determines if the series converges or not. We can always add a finite number of real numbers and the result will be a finite real number, but when we attempt to somehow “add” an infinite number of terms, in truth we can not but instead appeal to the well-defined partial sums $S_N$, and then use a limit argument to let $N \to \infty$.

This theorem is useful because many tests for convergence or divergence in fact allow us to ignore a finite number of terms in the series, to focus instead on the crucial “tail end” of the series.

Recall that if a series $\sum a_n$ is convergent, we say that the sequence of terms $a_n$ is summable. (It is also common to say the convergent series itself is “summable.”) It is helpful to have such vocabulary at our disposal to streamline later arguments.
10.3.1 Direct Comparison Test (DCT)

This test, like the Integral Test, is a test for series \( \sum a_n \) and \( \sum b_n \) of positive-term series. Here is its statement:

**Theorem 10.3.2 Direct Comparison Test (DCT) for Positive-Term Series.** For such series,

\[
0 \leq a_n \leq b_n, \quad \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges.} \quad (10.21)
\]

\[
0 \leq a_n \leq b_n, \quad \sum_{n=1}^{\infty} a_n \text{ diverges} \implies \sum_{n=1}^{\infty} b_n \text{ diverges.} \quad (10.22)
\]

Perhaps the easiest way to interpret this is as follows, keeping in mind we are only discussing series with nonnegative terms added:

- If the series with the larger terms converges, so must the series with the smaller terms.
- If the series with the smaller terms diverges, so must the series with the larger terms.

Again, to be precise, this is true assuming that all terms are nonnegative.\(^{16}\) To be even more precise, we should then also use “greater” for “larger,” and “lesser” for “smaller” above. See Figure 10.3.

**Proof:** First we show that \( \sum b_n \) converges implies \( \sum a_n \) converges. To show this, we note that if \( S_N \) is the \( N \)th partial sum of \( \sum a_n \), then clearly \( S_N \) is nondecreasing (since \( a_n \geq 0 \)), and furthermore

\[
S_N = a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_n \leq \sum_{n=1}^{\infty} b_n,
\]

\(^{16}\)It is common practice to bend the language somewhat to use “positive-term series” to refer to series \( \sum a_n \) where \( a_n \geq 0 \) for all \( n \). A less concise but more precise term would be “nonnegative-term series.” It is important that we do not have the terms changing sign along the series, which is a situation we will deal with Section 10.4.
so \( S_N \leq \sum b_n \), i.e., each \( S_N \) is bounded by the whole series \( \sum b_n \). Thus \( \{ S_N \}_{N=1}^\infty \)

is a bounded, nondecreasing sequence, and must therefore converge. This proves (10.21). In fact (10.22) is just its contrapositive so it is also then proven, but it is also interesting to prove it separately.

So suppose instead that \( \sum a_n \) diverges. Being a nonnegative-term series it must therefore diverge to infinity. So let \( S_N \) be the \( N \)th partial sum of \( \sum a_n \) and \( \mathcal{S}_N = b_1 + b_2 + \cdots + b_N \). Then

\[
\mathcal{S}_N = b_1 + b_2 + \cdots + b_N \geq a_1 + a_2 + \cdots + a_N = S_N \rightarrow \infty \quad \text{as} \quad N \rightarrow \infty,
\]

showing that \( \mathcal{S}_N \) diverges to infinity \((\mathcal{S}_N \geq S_N \rightarrow \infty)\), i.e., \( \sum b_n \) diverges (to infinity), q.e.d.

Of course the same is true if the two series start somewhere besides \( n = 1 \). Furthermore, in light of Theorem 10.3.1 (page 721), in fact we only require \( 0 \leq a_n \leq b_n \) to be true “eventually,” i.e., for all \( n \geq N \) where \( N \) is some finite number. (Here we use \( N \) differently than in the partial sums \( S_N \).)

**Corollary 10.3.1** Suppose for some \( N \) we have \( n \geq N \Rightarrow 0 \leq a_n \leq b_n \). Then (10.21) and (10.22) still hold.

**Example 10.3.1** Consider the series \( \sum_{n=1}^\infty \frac{1}{\ln n + n^{3/2}} \). We note that for \( n \geq 1 \) we have \( \ln n \geq 0 \), so the denominator is greater than or equal to \( n^{3/2} \), and of course \( \sum \frac{1}{n^{3/2}} \) converges.

\[
0 < n^{3/2} \leq \ln n + n^{3/2} \quad \Rightarrow \quad 0 < \frac{1}{\ln n + n^{3/2}} \leq \frac{1}{n^{3/2}} \quad \text{summable}
\]

\[
\therefore \sum_{n=1}^\infty \frac{1}{n^{3/2}} \text{ converges} \quad \Rightarrow \quad \sum_{n=1}^\infty \frac{1}{\ln n + n^{3/2}} \text{ converges.}
\]

(The first inequalities “\( 0 < \)” are just to indicate that all terms are positive, so the Direct Comparison Test, DCT, applies.) Since \( \sum \frac{1}{n^{3/2}} \) converges \((p\text{-series, } p = 3/2 > 1)\), we can conclude by the DCT that \( \sum \frac{1}{\ln n + n^{3/2}} \) also converges.

Note that we used the fact that, when all terms are positive, a larger denominator implies a smaller fraction.\(^{17}\) We also used that \( \ln n \geq 0 \) for \( n \geq 1 \) (though “eventually”, i.e., for all \( n \geq N \) for some \( N \), is enough).

**Example 10.3.2** Consider \( \sum_{n=1}^\infty \frac{\ln n}{n} \). While we could use an Integral Test on this series to determine convergence or divergence,\(^{18}\) it will be faster to note that, for large enough \( n \) (in

\(^{17}\)Recall that if \( A, B, C, D > 0 \), then

\[
A < B \quad \Rightarrow \quad \frac{A}{C} < \frac{B}{C}, \quad \text{(larger numerator \quad \Rightarrow \quad larger fraction)}
\]

\[
C < D \quad \Rightarrow \quad \frac{A}{C} > \frac{A}{D}, \quad \text{(larger denominator \quad \Rightarrow \quad smaller fraction)}
\]

\(^{18}\)We might also consider the NTTFD, Theorem 10.2.2 (page 715), but it does not apply since

\[
\lim_{n \to \infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \quad \text{LHR} \quad \lim_{n \to \infty} \frac{1}{n} = 0.
\]
particular, \( n > e^1 \) so for integers, \( n \geq 3 \) we have

\[
0 < \frac{1}{n} < \frac{\ln n}{n}.
\]

\[
\therefore \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \implies \sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ diverges.}
\]

The DCT is particularly useful when we can not (or can not easily) integrate the respective function, as in our first example (Example 10.3.1, page 723).

Example 10.3.3 Discuss the convergence/divergence of \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + 1}} \).

Solution: We note that the terms in the series would be larger if the “+1” were absent in the denominators.

\[
0 \leq \frac{1}{\sqrt{n^5}} < \frac{1}{\sqrt{n^5 + 1}} = \frac{1}{n^{5/2}} \text{ summable}
\]

\[
\therefore \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \text{ converges} \implies \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + 1}} \text{ converges.}
\]

10.3.2 A Hierarchy of Functions and DCT

It is useful to note that different functions \( f(n) \) which grow to \( \infty \) as \( n \to \infty \) do so much faster than others.

Theorem 10.3.3 In the list below, any function \( f(n) \) listed to the left of another function \( g(n) \) will grow so much more slowly than \( g(n) \) that \( \lim_{n \to \infty} f(n)/g(n) = 0 \).

\[\ln(\ln n), \ln n, n^r(r > 0), n^s (s > r), a^n (a > 1), b^n (b > 1), n!, n^n.\]

This is not an exhaustive list, but offers some useful facts and intuition. It will take some effort and later methods to show why \( n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \) fits into the hierarchy where it does. It is somewhat more intuitive to see that \( n^n \) is properly placed, at least in comparison to \( n! \):

\[
\begin{align*}
0! &= 1 \text{ (by definition)} \\
1! &= 1 \\
2! &= 1 \cdot 2 = 2 \\
3! &= 1 \cdot 2 \cdot 3 = 6 \\
4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 4 \cdot 3! = 24 \\
5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120 \\
6! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6 \cdot 5! = 720 \\
7! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 7 \cdot 6! = 5040
\end{align*}
\]

so the NTTFD is inconclusive. However it is also not difficult to observe that

\[
\int_1^{\infty} \frac{\ln x}{x} \, dx = \lim_{\beta \to \infty} \int_1^{\beta} \frac{\ln x}{x} \, dx = \lim_{\beta \to \infty} \frac{1}{2} \left( \ln \beta \right)^2 = \lim_{\beta \to \infty} \left[ \frac{1}{2} \ln \beta \right]^2 = \infty,
\]

which proves \( \sum \frac{\ln \beta}{n} \) diverges, by the Integral Test.
10.3. COMPARISON TESTS

and so on. This hierarchy helps us to use the DCT to determine convergence or divergence of some series, and sometimes the NTTFD to determine divergence.

Example 10.3.4 Consider \( \sum_{n=2}^{\infty} \frac{1}{\ln n} \). We can direct-compare this to \( \sum_{n=2}^{\infty} \frac{1}{n} \). (Note why we can not start at \( n = 1 \).) For larger enough \( n \) (actually for all \( n > 0 \)), we have \( \ln n < n \) (proven by the fact that \( (\ln n)/n \to 0 \), see original limit in Footnote 18, page 724), and so for large enough \( n \) we have

\[
0 < \frac{1}{n} < \frac{1}{\ln n} \quad \text{(not summable)}
\]

\[
\therefore \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges} \implies \sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges}.
\]

Example 10.3.5 Consider \( \sum_{n=1}^{\infty} \frac{1}{n!} \). Since for instance \( n! > n^2 \) for large enough \( n \) (in particular, \( n \geq 4 \)), we can write

\[
0 < \frac{1}{n!} < \frac{1}{n^2} \quad \text{(summable)}
\]

\[
\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \implies \sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges}.
\]

Note that not all comparisons are useful. For instance, it is true that \( 1/n! < 1/n \), but \( 1/n \) is not summable, i.e., \( \sum \frac{1}{n} \) diverges, so that comparison does not let us conclude convergence or divergence of the series \( \sum \frac{1}{n!} \):

\[
0 < \frac{1}{n!} < \frac{1}{n} \quad \text{(No Conclusion Possible!)}
\]

The problem is that we are using a series whose sums approach infinity, which is not a useful, or even proper, “bound” for anything. Any series we find, convergent or otherwise, can have its terms bounded by those of a divergent series. (Just take \( b_n = a_n + 1 \) or \( b_n = a_n + n \) or similar.) Thus the inequality above is indeterminate as far as determining in and of itself the convergence or divergence of the series of the terms on the left of the inequality. Similarly, knowing the terms of a series are larger than those of a convergent series is useless.

\[
0 < \frac{1}{n^2} < \frac{1}{n} \quad \text{(No Conclusion Possible!)}
\]

In fact, note that \( 0 < \frac{1}{n!n} < \frac{1}{n} \), and \( 0 < \frac{1}{n!(\ln n)^2} < \frac{1}{n} \), but \( \sum \frac{1}{n}(\ln n)^2 \) diverges while \( \sum \frac{1}{n\ln n} \) converges, both by routine applications of the Integral Test, so knowing that a series’ terms are smaller than those of \( \sum \frac{1}{n} \) (or any other positive-term divergent series) does not guarantee convergence or divergence.
Indeed, there are series that converge and series that diverge whose terms are smaller than those in the harmonic series $\sum \frac{1}{n}$. However there are no convergent series with terms greater than the harmonic series.

Note also we could have used the hierarchy of functions to conclude $\sum \frac{1}{n}$ converges, because $\frac{1}{n!} < \frac{1}{2^n}$, and $\sum (1/2)^n$ converges (geometric, $|r| = 1/2 < 1$), also giving us $\sum \frac{1}{n}$ converges by the DCT.

**Example 10.3.6** Consider $\sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3 + 5}}$. Since $0 \leq \cos^2 n \leq 1$, we can write

$$0 \leq \frac{\cos^2 n}{\sqrt{n^3 + 5}} < \frac{1}{\sqrt{n^3 + 5}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}},$$

so $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\cos^2 n}{\sqrt{n^3 + 5}}$ converges.

**Example 10.3.7** Consider $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$. Note that

$$0 < \frac{n}{n^4 + 1} < \frac{n}{n^4} = \frac{1}{n^3},$$

so $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$ converges.

### 10.3.3 Limit Comparison Test (LCT)

At times, the DCT requires a bit more cleverness than necessary. For instance, if we take the previous example and make minor changes, considering instead $\sum_{n=1}^{\infty} \frac{n}{n^4 - 1}$, our intuition is that for large enough $n$, the terms in this series should be very similar to those of $\sum_{n=2}^{\infty} \frac{n}{n^4 - 1}$, since the difference in the denominators becomes less and less significant for large $n$. So our intuition is that this series probably converges as well, but finding a formula for $S_N$, or a series which has larger terms but still converges, or even using an integral test, can be difficult (or in the case of finding a formula for $S_N$, perhaps impossible). One comparison that does work—INCLUDED here just for completeness—is the following:

$$0 < \frac{n}{n^4 - 1} = \frac{n^2}{n(n^4 - 1)} < \frac{2(n^2 - 1)}{n(n^4 - 1)} = \frac{2}{n(n^2 + 1)} < \frac{2}{n^3},$$

However the first equality does not immediately seem well motivated, and the second inequality is not so obvious unless perhaps we write out several terms of each to see the pattern emerging.

And yet it seems like we should be able to argue that $\sum_{n=1}^{\infty} \frac{n}{n^4 - 1}$ converges we can conclude that $\sum_{n=1}^{\infty} \frac{n}{n^4 - 1}$ also converges. Indeed we can, if we can appropriately quantify what we mean by $\frac{n}{n^4 - 1} \approx \frac{1}{n^3}$, according to the following theorem, if we can appropriately quantify what we mean by $\frac{n}{n^4 - 1} \approx \frac{1}{n^3}$.
Theorem 10.3.4 Limit Comparison Test (LCT). Suppose $a_n \geq 0, b_n > 0$ for large enough $n$, and that $\lim_{n \to \infty} \frac{a_n}{b_n} = L \in [0, \infty]$.

1. If $L \in (0, \infty)$, then
   \[ \sum a_n \text{ converges} \iff \sum b_n \text{ converges}, \text{ or equivalently} \]
   \[ \sum a_n \text{ diverges} \iff \sum b_n \text{ diverges}. \]

2. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

3. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

The idea of the theorem can be more casually described as follows (think $a_n \approx L \cdot b_n$):

1. For large $n$, we have $a_n \approx L \cdot b_n$, and so if $L \in (0, \infty)$, both series converge or both diverge, since the “tail end” terms of one series are approximately a constant multiple ($L \neq 0, \infty$) of the “tail end” terms of the other;

2. if $L = \infty$, then $a_n$ is a larger and larger multiple of $b_n$, so if $\sum b_n$ diverges, so does $\sum a_n$ (If $\sum a_n$ diverges or $\sum b_n$ converges, this is indeterminate regarding convergence or divergence of the other);

3. if $L = 0$, then $a_n$ is a smaller and smaller multiple of $b_n$, so if $\sum b_n$ converges, so does $\sum a_n$. (If $\sum a_n$ converges or $\sum b_n$ diverges, this is indeterminate regarding convergence or divergence of the other.)

A proof would actually rely on the DCT, by showing that we can eventually make the correct comparison with an appropriate series to give us the conclusion. Without going through every case, we look at the proof that if $\sum b_n$ converges, and $a_n/b_n \to L \in (0, \infty)$, then we must conclude $\sum a_n$ converges.

\textbf{Proof:} Assume $\lim_{n \to \infty} \frac{a_n}{b_n} = L \in (0, \infty)$, and that $\sum b_n$ converges. Then for large enough $n$, we have $a_n/b_n \approx L$, so $a_n/b_n \in (L - \frac{1}{2}L, L + \frac{1}{2}L) = (\frac{1}{2}L, \frac{3}{2}L)$. Thus for large enough $n$ we have $0 < a_n < \frac{3}{2} \cdot b_n$, so we can direct-compare $\sum a_n$ to $\sum \frac{3}{2} \cdot b_n$ (which converges, since $\sum b_n$ does):

\[ \sum b_n \text{ converges} \implies \frac{3}{2} \cdot \sum b_n = \sum \left( \frac{3}{2} \cdot b_n \right) \text{ converges} \implies \sum a_n \text{ converges,} \]

by the DCT. The other cases are proven similarly, with modifications.

The case proved above is the most commonly referenced case, and so we make the following definition:

\textbf{Definition 10.3.1} For two positive-term series $\sum a_n$ and $\sum b_n$, we call the two series limit-comparable if and only if

$\lim_{n \to \infty} \frac{a_n}{b_n} = L \in (0, \infty)$. 

\[ ^{19}\text{Technically, we have not defined what it means for } \lim_{n \to \infty} \frac{a_n}{b_n} = \infty, \text{ so we mean that the sequence } \{a_n/b_n\} \text{ diverges to infinity. For the curious reader, a definition of } c_n \to \infty \text{ would be } (\forall M > 0)(\exists N)(n > N \implies c_n > M). \]
Note we omit the cases \( L \in \{0, \infty\} \). With this definition, part of the Limit Comparison Test (LCT) can now be phrased:

Two limit-comparable series will both converge, or both diverge.

**Example 10.3.8** Consider \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}} \). We could use an integral test, but that would require either a complicated formula or trigonometric substitution to find the relevant antiderivative. The naïve Direct Comparison Test would have us note that \( 1/\sqrt{n^2 + 1} < 1/n \), but \( \sum (1/n) \) is not summable, so \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}} \) being a "smaller" series than a divergent series is inconclusive.

However, for large enough \( n \), it seems \( 1/\sqrt{n^2 + 1} \approx 1/\sqrt{n^2} = 1/n \), so the series seems similar to the harmonic series \( \sum (1/n) \), so we verify that the two are limit-comparable:

\[
\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n^2 + 1}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 1}} \approx \frac{n}{n} = 1 \in (0, \infty).
\]

This shows that the original series is limit-comparable to \( \sum 1/n \), which diverges, so we can conclude \( \sum \frac{1}{\sqrt{n^2 + 1}} \) also diverges.

**Example 10.3.9** Consider \( \sum_{n=1}^{\infty} \frac{2n + 1}{n^3 + 5n^2 + 6n} \). While an integral test is do-able, as is a DCT argument, it seems more natural to note that \( \frac{2n + 1}{n^3 + 5n^2 + 6n} \approx \frac{2n}{n^3} = \frac{2}{n^2} \) for large \( n \). We can limit-compare the series to \( \sum \frac{2}{n^2} \), or even \( \sum \frac{1}{n^2} \). For this example we will do the latter.

\[
\lim_{n \to \infty} \frac{\frac{2n + 1}{n^3 + 5n^2 + 6n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{(2n + 1)(n^2)}{n^3 + 5n^2 + 6n} = \lim_{n \to \infty} \frac{2n^3 + n^2}{n^3 + 5n^2 + 6n} = \lim_{n \to \infty} \frac{2 \cdot 1}{1 \cdot 1 + 0} = 2 \in (0, \infty).
\]

Our original series is limit-comparable to \( \sum \frac{1}{n^2} \), which converges, and so therefore must our original series.

In the conclusive cases where \( (a_n)/(b_n) \to L \in \{0, \infty\} \), there is usually a DCT argument that would also get us our result. For instance, if we look at \( \sum \frac{\ln n}{n} \), for the DCT we can show that \( \frac{\ln n}{n} \geq \frac{1}{n} \), and since \( \sum \frac{1}{n} \) diverges we know the same is true of \( \sum \frac{\ln n}{n} \). But if we instead limit-compare the two series, we get

\[
\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \ln n = \infty,
\]

and since \( \sum \frac{1}{n} \) diverges we can safely say that so does \( \sum \frac{\ln n}{n} \). (Recall the idea is that \( \frac{\ln n}{n} \) is a larger and larger multiple of \( \frac{1}{n} \), which is not summable.)

**Example 10.3.10** Consider \( \sum_{n=1}^{\infty} \frac{\ln n}{n} e^{-n} \). For this we can use either DCT or LCT.

For the Direct Comparison Test, we note that since \( \ln n < n \) for large \( n \), we have \( \frac{\ln n}{n} e^{-n} < 1 \cdot e^{-n} = \left(\frac{1}{e}\right)^n \), which is summable (geometric series, \( |r| = \frac{1}{e} < 1 \)). Hence so is \( \sum \frac{\ln n}{n} e^{-n} \).
10.3. COMPARISON TESTS

For the Limit Comparison Test, we limit-compare the original series to $\sum e^{-n}$, which again converges (see above). So we compute

$$
\lim_{n \to \infty} \frac{\ln n e^{-n}}{e^{-n}} = \lim_{n \to \infty} \frac{\ln n}{n} = 0,
$$

which we get from either one l'Hôpital's Rule step or from our hierarchy of functions. Since $\sum e^{-n}$ converges, and the limit above is zero, so does $\sum \frac{\ln n e^{-n}}{n}$ converge by a special case ($L = 0$) of the LCT. (Recall the idea there is that the given series terms are smaller and smaller multiples of the convergent series $\sum e^{-n}$.)

The limit comparison test can sometimes produce limits which are challenging to compute. Often such examples would be easier with a later technique, as in the next example (which would be more appropriate for a Ratio Test introduced in Section 10.5), but ultimately such tests rely for their proofs on these more primitive tests.

Example 10.3.11Consider the series $\sum_{n=1}^{\infty} \frac{n}{\ln n} e^{-n}$.

This has three functions from our hierarchy, listed here in the order they appear in the hierarchy: $\ln n$, $n$ and $e^n$ (in a denominator).

Now $\frac{n}{\ln n} e^{-n}$ may indeed shrink, but not as quickly as $e^{-n}$, since $\frac{n}{\ln n} \to \infty$. Note that $e \in (2, 3)$, so $1/e \in \left(\frac{1}{3}, \frac{1}{2}\right)$, so we will compare the original series to $\sum \left(\frac{1}{2}\right)^n$, whose terms shrink more slowly than $\sum e^{-n}$, so we will see how $\sum 2^{-n}$ compares to our original series in the limit.

$$
\lim_{n \to \infty} \frac{n}{\ln n} e^{-n} = \lim_{n \to \infty} \frac{n}{\ln n} \left(\frac{2}{e}\right)^n
= \lim_{n \to \infty} \frac{n}{(e/2)^n} \cdot \frac{1}{\ln n} = 0.
$$

The last computation is based on the hierarchy of functions again, with the polynomial power $n^1$ divided by an exponential $(e/2)^n$, since $e/2 > 1$. A LHR argument could also work. Noting that $2/e < 1 \implies \ln(2/e) < 0$, this limit can also be arrived at by defining $y = \frac{n}{\ln n} \left(\frac{2}{e}\right)^n$, and finding $\lim_{n \to \infty} y = -\infty$, implying $y \to 0$:

$$
\lim_{n \to \infty} \ln y = \lim_{n \to \infty} \left[ n \ln \frac{2}{e} + \ln n - \ln(\ln n) \right] = -\infty \implies \lim_{n \to \infty} y = 0.
$$

While this limit on the left looks like an indeterminate case where we need to combine the expressions into one (large) fraction and use l'Hôpital's Rule, in fact if we look at the hierarchy of functions, Theorem 10.3.3 (page 724), we see that the first-degree polynomial term $(\ln \frac{2}{e})n$ grows much faster in size than $\ln n$, and for that matter $\ln(\ln n)$, so its effect will dominate as $n \to \infty$. Furthermore, the limit above will be $-\infty$, since the coefficient of the dominating $n^1$-term is $\ln \frac{2}{e} < 0$. (Again, $2 < e \implies \frac{2}{e} < 1 \implies \ln \frac{2}{e} < 0$, and in fact $\ln \frac{2}{e} \approx -0.3068528 < 0$.)

However we arrive at the limit, what we have is that

$$
\lim_{n \to \infty} \frac{n}{\ln n} e^{-n} = 0,
$$

and since $\sum \left(\frac{1}{2}\right)^n$ converges, so does $\sum \frac{n}{\ln n} e^{-n}$. (Again the idea was $\frac{n}{\ln n} e^{-n}$ is a shrinking multiple of $(1/2)^n$, which is summable.)
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While $\sum \frac{n}{\ln n} e^{-n}$ had terms bigger than the geometric series $\sum e^{-n}$, its terms were actually smaller than another geometric series $\sum \left(\frac{1}{2}\right)^n$, whose terms were slightly larger than those of $\sum e^{-n}$.

The argument in the previous example was quite sophisticated, and indeed was not really necessary, after we develop the tools of Section 10.5, but much can be done with these more primitive tests. (Also, as mentioned earlier, the more sophisticated tests ultimately rely on DCT for their proofs, so in principle any series using those tests could have convergence or divergence proven by DCT, though it is often much easier to use the computational machinery built into the later tests.)

10.3.4 Summary

The two tests developed here, namely DCT and LCT, rely on our being able to identify series to compare our given series to in a meaningful way to predict convergence or divergence of our given series, based upon convergence or divergence of a known series, typically a $p$-series or geometric series from before. This takes practice, because if we compare a series to another in an inconclusive (or indeterminate) way, we will be tempted to apply the two tests in invalid ways.

Briefly, the main theorems for positive-term series were

(DCT) \[
\begin{align*}
0 \leq a_n &\leq b_n, & \sum b_n \text{ converges} &\implies \sum a_n \text{ converges}, \\
0 \leq a_n &\leq b_n, & \sum a_n \text{ diverges} &\implies \sum b_n \text{ diverges},
\end{align*}
\]

(LCT) \[
\begin{align*}
\lim_{n \to \infty} \frac{a_n}{b_n} = L \in (0, \infty) &\implies \sum a_n, \sum b_n \text{ both converge or both diverge}.
\end{align*}
\]

There were other cases of the LCT, namely $L = 0$ or $L = \infty$, but if we keep in mind that $a_n \approx L \cdot b_n$ for $n$ large, we can predict them as well.\(^20\)

---

\(^20\)We wrote down two cases, but each had contrapositives as well, so it can be rather confusing to list them all.
10.3. COMPARISON TESTS

10.3.5 Elementary Series Theorems

In order to take full advantage of the Direct and Limit Comparison Tests (DCT and LCT) above, it is best if we have some elementary results regarding series at our disposal. These are intuitive, though as always some care must be taken to be clear about what they do and do not say.

**Theorem 10.3.5** Suppose \( k \neq 0 \). Then \( \sum a_n \) converges if and only if \( \sum (k \cdot a_n) \) converges, and furthermore

\[
\sum (k \cdot a_n) = k \cdot \sum a_n.
\]  

**Proof:** This is just the observation that, assuming \( n \) starts at 1 in the sum \( \sum (k \cdot a_n) \), we have

\[
\sum_{n=1}^{N} (k \cdot a_n) = ka_1 + ka_2 + \cdots + ka_N = k(a_1 + a_2 + \cdots + a_N).
\]

**Example 10.3.12** Discuss the convergence or divergence of \( \sum \frac{1}{n^2} \).

**Solution:** For each of these, we simply “factor” the multiplicative constants:

- \( \sum \frac{1}{n} \) diverges (since \( \sum \frac{1}{n} \) diverges).
- \( \sum \frac{2}{n} = 2 \sum \frac{1}{n} \) converges (since \( \sum \frac{1}{n} \) converges).

Now either of the above examples could have been solved using the LCT above, since the first is limit-comparable to \( \sum \frac{1}{n} \), and the other to \( \sum \frac{1}{n^2} \), but the arguments above are much more efficient for these examples.

We next look at combinations of two series, in some cases.

**Theorem 10.3.6** Given two series \( \sum a_n \) and \( \sum b_n \).

1. If \( \sum a_n \) and \( \sum b_n \) both converge, then

\[
\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n.
\]  

Furthermore, if one of the series, \( \sum a_n \) or \( \sum b_n \) converges but the other diverges, then \( \sum (a_n \pm b_n) \) diverges.

The idea of the proof is pretty simple. We start with the case where both converge and we are adding. Suppose \( S_N \) is the partial sum of the series \( \sum (a_n + b_n) \). Then

\[
S_N = a_1 + b_1 + a_2 + b_2 + \cdots + a_N + b_N = \underbrace{a_1 + a_2 + \cdots + a_N}_\sum a_n + \underbrace{b_1 + b_2 + \cdots + b_N}_\sum b_n.
\]

Thus \( \sum (a_n + b_n) = \sum a_n + \sum b_n \). From the limit argument above as well, we can see that if one of the series, say \( \sum a_n \) converges but \( \sum b_n \) diverges, it is necessary that \( \sum (a_n + b_n) \) diverges. In that case we wouldn’t actually write (10.26), because it makes no sense to have an equation where a divergent series is an expression added to another quantity. Similarly one should not write, for instance

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \quad \text{(FALSE!)}
\]
because the series on the left actually converges (as a telescop ing series, or we can limit or direct compare it to $\sum \frac{1}{n^2}$ since it can be combined to give $\sum \frac{1}{n(n+1)}$), where the two on the right do not.

The upshot of all this is that (10.25) and (10.26) are both true and make perfect sense, if the underlying series converge. When they don't, there may or may not be something useful to say about the combined sum.
10.4. Alternating Series and Absolute Convergence

Unlike the previous two sections on positive-term series, in this section we look at series \( \sum a_n \) which alternate signs, between positive and negative terms. Such series are called, naturally enough, alternating series. It turns out that if the terms alternate, and their sizes shrink monotonically to zero, then that is enough for the series to be known to converge. Basically, even if the terms do not shrink to zero quickly, there is a recurring partial cancellation, somewhat like we saw in telescoping series, though here we are much less likely be able to find a simple formula for the partial sums whose limits we can compute.

Some alternating series rely on the alternation for convergence, and are thus called conditionally convergent, where other alternating series have the terms shrink fast enough that the alternation is not in fact necessary to ensure convergence. The latter series are called absolutely convergent, for reasons that will become clear later in this section.

10.4.1 Alternating Series Test (AST)

The main result in this section is the following (see Figure 10.4).

**Theorem 10.4.1** Suppose \( \sum_{n=1}^{\infty} a_n \) satisfies the following three conditions:

1. \( \text{The terms of the series alternate signs \( (+, -, +, -, +, \cdots \) i.e.,} \quad \frac{a_{n+1}}{a_n} < 0 \text{ for all } n = 1, 2, 3, \cdots \)

2. \( |a_1| \geq |a_2| \geq |a_3| \geq \cdots \) i.e., \( \{ |a_n| \}_{n=1}^{\infty} \) is a decreasing sequence.

3. \( \lim_{n \to \infty} |a_n| = 0 \).
Rephrased, we suppose the terms of \( \{a_n\} \) alternate signs and shrink in absolute size monotonically to zero. If this is the case, then \( \sum a_n \) converges.

Recall that \( \sum a_n \) converges means that the sequence \( \{S_N\}_{N=1}^{\infty} = \left( \sum_{n=1}^{N} a_n \right)_{N=1}^{\infty} \) of partial sums converges to some finite number \( S \), i.e., \( S_N \to S \in \mathbb{R} \) as \( N \to \infty \). This is partially illustrated in Figure 10.4, page 733, where we see the partial sums form a sequence of terms \( S_N \) which oscillate left and right on the number line, but between tighter and tighter confines.

It should also be noted that it is only necessary for the series’ terms to show the alternation and shrinking-to-zero (monotonically) hypotheses of the AST “eventually,” i.e., for all \( n > N \), some \( N \geq 1 \). For simplicity many texts assume \( |a_1| > |a_2| > |a_3| > \cdots \) and \( |a_n| \to 0 \), but the weaker “\( \geq \)” suffices. Note that we can summarize quickly (2) and (3) above by writing

\[
|a_1| \geq |a_2| \geq |a_3| \geq \cdots \to 0.
\]

Along with the alternation, this implies convergence.\(^{21}\)

It is also interesting to note that if \( |a_n| \neq 0 \), then \( a_n \neq 0 \) implying \( \sum a_n \) diverges (NTTFD, Section 10.2).

**Example 10.4.1** Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \).

If we write out a few terms of this series, we get

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.
\]

Clearly the series’ terms alternate signs, and shrink monotonically in absolute value to zero:

\[
\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{n} \right| = \lim_{n \to \infty} \frac{|(-1)^{n+1}|}{|n|} = \lim_{n \to \infty} \frac{1}{n} = 0.
\]

If we wish to check for monotonicity of \( |a_n| \), we can note that the denominator is obviously increasing monotonically, and the numerator is constant, so the fractions which are \( |a_n| = \frac{n}{n} \) are clearly decreasing (to zero) monotonically. We could also note that if \( f(x) = \frac{1}{x} \), then \( f'(x) = -1/x^2 < 0 \) for \( x \geq 1 \) (or even all \( x \neq 0 \)), so \( |a_n| = f(n) \) is clearly decreasing monotonically for \( n \geq 1 \).

With all this, we know the series converges by the AST.

Usually a short inspection assures us that the terms in the series shrink in absolute size monotonically to zero, though sometimes sophisticated arguments are required to make this clear.

Note that the convergence would still apply if the alternation of signs began with a negative term, as in

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots.
\]

\(^{21}\)Some texts define an alternating series by assuming \( \{a_n\} \) is a positive-term series, and then considering the series \( \sum (-1)^n a_n \) or \( \sum (-1)^{n+1} a_n \) or similar series. This arguably has some advantage later, but it is minor, if existent at all.
In fact the series above would be the additive inverse (negative) of the series in the previous example above (so they would both converge or both diverge). Furthermore, it should be pointed out that this new series can also be written
\[\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots.\]

One interesting aspect of the convergent alternating series is that we can estimate \(|S - S_N|\) easily from an illustration such as Figure 10.4 at the beginning of the section (page 733).

**Theorem 10.4.2** Suppose \(\sum a_n\) is an alternating series which satisfies the hypotheses (1)–(3) of the AST, Theorem 10.4.1, page 734. Then
\[|S_N - S| \leq |a_{N+1}|.\]

Furthermore, if we can replace the inequality \(\geq\) with \(\succ\) in the hypotheses of the AST, we can replace the inequality \(\leq\) with \(\prec\) in (10.27).

Indeed, the distance between \(S_N\) and \(S\) is always (for such series) no more than \(|a_{N+1}|\) because by adding \(a_{N+1}\) to \(S_N\) we “overshoot” \(S\) in arriving at \(S_{N+1}\).

For a simple application of (10.4.2), suppose we wish to approximate the series \(S = \sum (-1)^{n+1}/n\) by taking \(S_N\) for \(N\) large enough that \(|S_N - S| < 0.001\). Then we can use this estimate to find \(N\) large enough to be sure \(S_N\) is indeed within 0.001 of the full series \(S\). We do this by inserting the inequality (10.27) within \(|S - S_N| < 0.001\):
\[|S - S_N| < |a_{N+1}| < 0.001\text{ \text{inserted}}\]
\[\implies \frac{1}{N+1} < 0.001\]
\[\implies 1000 < N + 1\]
\[\implies 1000 < N + 1\]
\[\implies 999 < N.\]

Thus we need \(N > 999\), or \(N \geq 1000\) to guarantee by (10.27) we have \(|S - S_N| < 0.001\).

Other series which we can quickly see converge by the AST follow:
\[\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}, \quad \sum_{n=1}^{\infty} (-1)^n \sin \left(\frac{1}{n}\right).\]

In fact the last series may warrant a check:
\[f(x) = \sin \frac{1}{x} \implies f'(x) = \left(\cos \frac{1}{x}\right) \cdot \frac{1}{x^2} < 0\]
for large enough \(x\) that \(\cos \frac{1}{x} \approx \cos 0 = 1 > 0\). Furthermore, \(|(-1)^n \sin \frac{1}{n}| \to |\sin 0| = 0\) as \(n \to \infty\).

On the other hand, we must be careful to note that alternation alone is not enough to conclude a series converges.

\[22\text{Using methods from the next chapter, we can in fact show that } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \approx 0.693147181. \text{ To guarantee this kind of accuracy (±0.000000001 = ±10^{-9}) using (10.27) we would, for this series, need to sum } N = 10^9 \text{ terms, which would require some care and skill even for a computerized computation.}\]
Example 10.4.2 Consider \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \).

Here we have alternation, but
\[
\lim_{n \to \infty} \left| \frac{(-1)^n}{n+1} \right| = \lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0,
\]
and so the series diverges. The last limit can be computed using either algebra (factor \( n \) from the numerator and denominator) or l'Hôpital's Rule.

Similarly, \( \sum \frac{(-1)^n}{n \ln n} \) will not converge, since \(|(-1)^n \ln n| = n/\ln n \to \infty \) as \( n \to \infty \). Ultimately it is the NTTFD that lets us conclude divergence, since \(|a_n| \to 0 \iff a_n \to 0 \).

10.4.2 Absolutely and Conditionally Convergent Series

Here we point out that there is a convergence which is stronger (more stringent) than our previous definition that \( S_N \to S \in \mathbb{R} \). Before arriving at a definition of this stronger convergence criterion, however, we first look more closely at two similar but crucially different alternating series:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{10.28}
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots \tag{10.29}
\]

Note that if we remove the alternation from the first series (10.28), it becomes the harmonic series and therefore diverges. That is not the case with the second series, which if we remove the alternation we get \( \sum \frac{1}{n^2} \), which is a \( p \)-series with \( p = 2 > 1 \) so it converges.

Putting this another way, the first series relies on the alternation to converge; the second series has terms which shrink fast enough that if we did not allow alternation (by removing the factor \((−1)^{n+1}\) or other methods), we still get a finite number for our “infinite series.”

The manner in which we detect if the terms shrink fast enough that they do not require alternation is to insert absolute values around each term in the series, which makes each term nonnegative and therefore eliminates the partial cancellation which the AST relied upon for the intuition behind that theorem.

So we note that
\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \text{ (diverges)},
\]
\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots \text{ (converges)}.
\]

So somehow the convergence of series (10.29) seems stronger than that of series (10.28), but we must make this more precise, as we do below.

**Definition 10.4.1** We call a series \( \sum a_n \) absolutely convergent if and only if \( \sum |a_n| \) converges.

---

\(^{23}\)This is not true for limits other than zero; for other limits, we only have \( \iff \). For instance, \( \lim_{n \to \infty} a_n = -3 \iff \lim_{n \to \infty} |a_n| = 3 \). In fact \( \lim_{n \to \infty} |a_n| = 3 \) is even true for the divergent sequence \( \{3(-1)^n\} \). However, \( |x| = 0 \iff x = 0 \), while \( |x| = 3 \iff x = 3 \).
10.4. ALTERNATING SERIES AND ABSOLUTE CONVERGENCE

So the term “absolutely” refers to the absolute values we inserted. However, while “absolutely convergent” seems in this context to mean the series with absolute values inserted does converge, in another context “absolutely convergent” seems to indicate a type of magnified “regular convergence” (in the sense \( S_N \to S \in \mathbb{R} \)), with “absolutely” as an adjective. In fact, both interpretations are correct, the second one following from the theorem below.

**Theorem 10.4.3** Suppose \( \sum a_n \) is absolutely convergent, i.e., \( \sum |a_n| \) converges. Then \( \sum a_n \) also converges, in the sense that its partial sums form a convergent sequence \( \{ S_N \} \).

In other words, absolute convergence implies regular convergence:

\[
\sum |a_n| \text{ converges } \implies \sum a_n \text{ converges.} \tag{10.30}
\]

We will not prove this, since it is more appropriate for a course in real analysis. However it should have the ring of truth.

**Example 10.4.3** Consider \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \).

While this series converges by the AST, we can also prove that it converges absolutely. In fact

\[
\left| \frac{(-1)^n}{n!} \right| = \frac{1}{n!} < \frac{1}{n^2}
\]

for large enough \( n \). Since \( \sum \frac{1}{n^2} \) converges, so must \( \sum \frac{1}{n!} \). Thus \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \) is absolutely convergent.

Note that the series in the above example is defined at \( n = 0 \) while \( \sum \frac{1}{n^2} \) is not, but we only need to be sure the “tail ends” of the series are direct-comparable. Note also that once we insert absolute values, we are back to the positive-term series tests, such as the Integral Test, DCT (used above), and LCT.

It should also be pointed out that absolute convergence and convergence are the same for positive-term series. In fact they are also the same for negative-term-only series as well. Only when we know there is some alternation do the two concepts differ.

Finally, if the series has some alternation of signs, then \( \sum |a_n| \neq \sum a_n \), and in fact \( |\sum a_n| < \sum |a_n| \) since (among other reasons) the former allows cancellation and the latter does not.\(^{24}\)

So we have series that converge absolutely, and series which converge but not absolutely, and series which diverge. For the second type we have another name to identify them more precisely:

**Definition 10.4.2** If \( \sum a_n \) converges but not absolutely, we call that series conditionally convergent.

In other words, \( \sum a_n \) is conditionally convergent iff \( \sum a_n \) converges but \( \sum |a_n| \) diverges. In such a case we note that the convergence of the original series must have been due to some alternations of sign,\(^{25}\) and if we remove the alternation by inserting absolute values around each term, the terms of the series do not shrink fast enough to be summable, so convergence is conditioned on the alternation.

---

\(^{24}\)It is always important to distinguish between convergence of a series and what it actually converges to, which are two different questions. Many of the convergence tests do not pretend knowledge of the actual value of the series, though some hints regarding its value may be present in the logic of a given test, or its particular application.

\(^{25}\)though not necessarily a consistent \(+ - + - - + \cdots\), since other patterns may similarly account for it, such as \(+ - + + - - - + + \cdots\) or similar.
Thus $\sum \frac{(-1)^{n+1}}{n}$ converges (AST), but not absolutely, and is thus conditionally convergent.

So to restate a fact mentioned earlier, any given series is either absolutely convergent, conditionally convergent, or divergent. The union of the first two types is what we simply call convergence ($S_N \to S \in \mathbb{R}$). Next we list a few quick examples, some already considered, but which help put these concepts into context.

<table>
<thead>
<tr>
<th>series</th>
<th>converges?</th>
<th>absolutely?</th>
<th>conditionally?</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum \frac{(-1)^n}{n^2}$</td>
<td>yes</td>
<td>yes</td>
<td>—</td>
<td>$\sum \frac{(-1)^n}{n^2} = \sum \frac{1}{n^2}$</td>
</tr>
<tr>
<td>$\sum \frac{(-1)^n}{n}$</td>
<td>yes</td>
<td>—</td>
<td>yes</td>
<td>$\sum \frac{(-1)^n}{n} = \sum \frac{1}{n}$</td>
</tr>
<tr>
<td>$\sum \frac{(-1)^n \ln n}{n}$</td>
<td>yes</td>
<td>—</td>
<td>yes</td>
<td>$\sum \frac{(-1)^n \ln n}{n} = \sum \frac{\ln n}{n}$</td>
</tr>
<tr>
<td>$\sum \frac{(-1)^n}{n+1} \ln n$</td>
<td>no</td>
<td>—</td>
<td>—</td>
<td>$\frac{(-1)^n}{n+1} = \frac{n}{n+1} \to 1 \neq 0$</td>
</tr>
<tr>
<td>$\sum \frac{(-2)^n}{3^n+1}$</td>
<td>yes</td>
<td>yes</td>
<td>—</td>
<td>$\sum \frac{(-2)^n}{3^n+1} = \frac{1}{3} \sum \left(\frac{2}{3}\right)^n$</td>
</tr>
</tbody>
</table>

The series we get when we insert absolute values can be tested using, respectively, $p$-series ($p = 2$), $p$-series ($p = 1$), DCT or the Integral Test, NTTFD, and geometric series convergence criteria.

Note that absolute convergence and conditional convergence both imply “old-fashioned” convergence ($S_N \to S \in \mathbb{R}$), but otherwise are mutually exclusive: a series can not be both absolutely convergent and conditionally convergent. One way to illustrate this is with a “possibility tree” like given below.

```
Converges? Absolutely?
  yes
  no

absolutely convergent
  yes
  no

conditionally convergent
  yes
  no

divergent
```

In previous sections, we only concerned ourselves with the first question in the tree, regarding convergence or divergence. Now we get more specific, and ask what kind of convergence. To be sure, if the terms we attempt to sum all have the same sign, then convergence and absolute convergence are the same. It is when the terms to add have nonconstant sign that we ask whether the series diverges, or converges only because of the alternation, or would have converged even without the alternation.

The next section contains two tests which can only detect absolute convergence, or divergence, or, in many cases, neither. Indeed, as in most of our series tests, at times they are conclusive and at other times they are inconclusive.

### 10.4.3 One Last Remark Concerning Absolute Convergence

One interesting aspect of an absolutely convergent sequence is that it does not matter what order we use to sum the terms, as long as all are summed in the limit. In fact we can even pick
out two or more “subseries” and sum them separately. So for instance, if $\sum |a_n|$ converges, then

$$\sum a_n = \sum a_{2n} + \sum a_{2n-1},$$

i.e., we can add the even and odd terms separately and get the same result. We can not do this with a conditionally convergent, alternating series since both of these “subseries” will diverge.

We will not prove this remark, but upon some reflection it should have a ring of truth. It is similar to how we accumulate the areas of a convergent improper integral, even if it has multiple “improper” endpoints.

In fact, an elementary homework exercise in senior-level analysis is to show that if you choose any real number $R \in \mathbb{R}$, then any conditionally convergent series can have its terms rearranged in such a way that the sum converges to $R$, in the sense that its new (after the rearrangement) partial sums $S_N$ do. (We can also rearrange the terms so the series diverges.) Thus the order in which we add terms matters very much in any conditionally convergent series. This is not the case with absolutely convergent series; order of addition in $\sum a_n$ is not an issue if $\sum |a_n|$ converges.

Example 10.4.4 Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

We cannot break this into two series of even and odd terms, because if we attempted to do so with this convergent series, we would get

$$\sum_{n \text{ odd}} \frac{(-1)^{n+1}}{n} + \sum_{n \text{ even}} \frac{(-1)^{n+1}}{n} = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{2n} \quad (\text{FALSE!})$$

The problem is that both series diverge to $\infty$. Furthermore, there are many orders we could rearrange these terms. For demonstration purposes, we can argue that one method leaves us with a sum greater than 1, and another with a sum less than $-1$. Here is how we can do that.

1. Add $1 + \frac{1}{3} + \frac{1}{5} = 1 + \frac{8}{15} > 1 + \frac{1}{2}$.
2. Add $-\frac{1}{2}$ to this previous sum: $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} > 1$.
3. Add $\frac{1}{7} + \frac{1}{9} + \cdots$ to the sum until it is greater than $1 + \frac{1}{2}$. We can do this since $\frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$ diverges.
4. Add $-\frac{1}{6}$, and the partial sum will still be greater than 1.
5. Add enough of the remaining odd terms of the original series until the sum is greater than $1 + \frac{1}{6}$.
6. Add $-\frac{1}{6}$. The sum is still greater than 1.
7. Continue with this pattern forever. The sums will continue to be greater than 1, and we can always add more terms from the tail-end of the (divergent) odd series to overcome the next term to be added in the even series, and the sums will continue to be greater than 1.

A similar process can be achieved to keep the sum less than $-1$, by choosing enough even terms to sum to a number less than $-2$, then adding 1, then adding enough of the remaining
even terms until the sum is a number less than $-1 - \frac{1}{3}$, and then adding $\frac{1}{3}$, and summing enough of the remaining even terms until the total sum is less than $-1 - \frac{1}{5}$, then adding $\frac{1}{5}$, and so on.

Both procedures give us a rearrangement of the terms of the original alternating series, but the new partial sums will remain forever greater than 1 in the first case, and less than $-1$ in the second. Both procedures ensure that every term in the original sum is included eventually in the new sum.

For an absolutely convergent series like $\sum \frac{(-1)^{n+1}}{n^2}$, the above process is impossible because no partial series will diverge (see Steps 3, 5 above), and any way we add all of the terms (eventually with some limit process) will result in the same sum.
10.5 Ratio and Root Tests

The Ratio and Root Tests introduced here detect if the “infinite tail-end” behavior of a given series is comparable to the behavior of a geometric series, and if so, with what ratio. For many interesting series, these tests can lead us to easily determine if the given series converges absolutely, or diverges. (Conditional convergence and its detection was addressed in Section 10.4.) Which test is used depends upon which computation is easier for a given series, usually as determined by the algebra involved in certain limit computations.

The proofs of the tests rely upon knowledge of geometric series and the Direct Comparison Test, so in principle any of the problems here can be computed using a Direct Comparison Test, or sometimes using other tests (such as the Integral Test), but the tests here will often be much easier to use, depending upon the series.

As with other techniques, there will be many important series for which the Ratio Test or Root Test is appropriate, and some other important series for which they are both inconclusive and therefore useless, so these tests are not replacements for the previous tests. For instance, the tests here are inappropriate for analyzing $p$-series, as we will see, but many other series will be difficult to conclusively analyze any other way besides using one of these two tests given in this section.

Note that if a geometric series converges, it does so absolutely, since it is in fact $|r| = |a_{n+1}|/|a_n|$ which is the crucial quantity: $\sum |\alpha r^n|$ converges if and only if $\sum |\alpha r^n|$ converges since the absolute values of the ratios of these two series are both $|r|$. In other words, applying the absolute value to each term would not change convergence (though it would certainly change the value of the series).

For the ratio and root tests, we will define a quantity $\rho$ (the lower-case Greek letter “rho”) for a given series, and this $\rho$ will mimic $|r|$ from geometric series. There are two ways we define $\rho$, but they are usually equal (at least for examples found here), though we decline to prove it for this discussion. What we will prove is that they are equal to $|r|$ for a geometric series. Furthermore, when $\rho < 1$ (i.e., $\rho \in [0,1)$) the series will converge, and do so absolutely; when $\rho > 1$ the series will diverge, and in fact a well-informed version of the NTTFD would apply but it might not be obvious because the limit involved in the NTTFD ($a_n \neq 0$) could be much harder to compute (or prove) than those that appear in the Ratio or Root Test.

Unfortunately, when $\rho = 1$ the tests will in fact be inconclusive, as we will show with examples, so there is not a perfect correlation with conclusions based upon $|r|$ from the geometric series, so we have to make note of that. What this tells us is that there is some room between the geometric series with $|r| = 1$ (which diverges), and series with $\rho = 1$, some of which converge and some of which diverge. In those cases we have to look to one of our previous techniques to attempt to find a conclusive test. As hinted previously, the $p$-series will all have $\rho = 1$, and of course some converge and others diverge. Many alternating series will also have $\rho = 1$, but some will converge (because $|a_n|$ shrinks to zero), while others will diverge (by NTTFD, since $|a_n| \neq 0$).

For the Ratio Test (RAT) we will define

$$\rho_{\text{Ratio}} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}.$$  

Note that for a geometric series $\sum \alpha r^n$, with $\alpha, r \neq 0$, this becomes

$$\rho_{\text{Ratio}} = \lim_{n \to \infty} \frac{|\alpha r^{n+1}|}{|\alpha r^n|} = \lim_{n \to \infty} |r| = |r|.$$
Thus \( \rho_{\text{Ratio}} = |r| \) for the case of a geometric series. For the Root Test (ROOT) we instead define
\[
\rho_{\text{Root}} = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{1/n}.
\]
For the geometric series this becomes
\[
\rho_{\text{Root}} = \lim_{n \to \infty} |\alpha r^n|^{1/n} = \lim_{n \to \infty} |\alpha|^{1/n} |(r|n|)^{1/n} = |\alpha|^{0} |r| = |r|.
\]
Rather than distinguish between \( \rho_{\text{Ratio}} \) and \( \rho_{\text{Root}} \), we will simply refer to both as \( \rho \), especially since they are usually the same number.

However, for each computation it should be clear from the context which definition of \( \rho \) is used.

If we calculate \( \rho \) for a series \( \sum a_n \), and \( \rho < 1 \), then we can interpret this to mean \( \sum |a_n| \) behaves very much like a geometric series with ratio \( \rho \), in the sense that they somehow converge similarly (and absolutely, but probably to different values). The same is true of series \( \sum a_n \) where \( \rho > 1 \), diverging the same way such a geometric series would. As mentioned before, unfortunately the tests below are not comprehensive. If \( \rho = 1 \) or does not exist, we cannot immediately decide from that fact alone whether or not the series converges (absolutely or otherwise), or diverges. However the tests are quite useful for numerous and important cases. These tests follow next.

**Theorem 10.5.1 Ratio Test (RAT):** Suppose for a series \( \sum a_n \) the limit
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]
exists, or is \( \infty \) (i.e., the sequence \( \{|a_{n+1}/a_n|\} \) diverges to \( \infty \)). Then
1. If \( \rho \in [0, 1) \), i.e., \( \rho < 1 \) then \( \sum a_n \) converges absolutely.
2. If \( \rho > 1 \) (including \( \rho = \infty \)), then \( \sum a_n \) diverges.
3. If \( \rho = 1 \) then this test is inconclusive (and some other test must be used).

**Theorem 10.5.2 Root Test (ROOT):** Suppose for a series \( \sum a_n \) the limit
\[
\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{1/n}
\]
exists, or is \( \infty \). Then
1. If \( \rho \in [0, 1) \), i.e., \( \rho < 1 \) then \( \sum a_n \) converges absolutely.
2. If \( \rho > 1 \) (including \( \rho = \infty \)), i.e., \( \rho > 1 \) then \( \sum a_n \) diverges.
3. If \( \rho = 1 \) then this test is inconclusive (and some other test must be used).

It is somewhat an art to decide which of the two tests (if any) is better suited for calculating \( \rho \). The first is better for factorials, and the second usually better if \( a_n \) is of the form \( (f(n))^n \), but there are exceptions. Of the two tests the Ratio Test is more often used, but there are certainly cases where the Root Test is closer to ideal, as in cases where \( |a_n|^{1/n} \) simplifies nicely.

\[26\]In fact, for most series found in calculus textbooks, \( \rho_{\text{Ratio}} = \rho_{\text{Root}} \), so both are just referred to as \( \rho \). However, if \( \sum a_n \) had infinitely many zero terms, or nearly-zero terms between larger terms (an oscillating sequence of some kind), the Ratio Test might be more problematic than the Root Test, but where both \( \rho \)’s are defined they will coincide.
Example 10.5.1 Consider \( \sum_{n=0}^{\infty} \frac{1}{n!} \). Using the Ratio Test, we compute \( \rho \). Since computations involving factorials will become more and more important in this and the next chapter, we will write out this particular computation in some extra detail.

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{\frac{(n+1)!}{n!}} = \lim_{n \to \infty} \frac{n!}{1} \cdot \frac{1}{(n+1)!} = \lim_{n \to \infty} \frac{n!}{(n+1)n!} = \lim_{n \to \infty} \frac{1}{n+1} \cdot \frac{1}{\infty} < 1.
\]

Therefore \( \sum_{n=0}^{\infty} \frac{1}{n!} \) converges (absolutely, but that is redundant here since all terms are positive).

We could have also used a Direct Comparison Test (DCT) along with our “Hierarchy of Functions” (Theorem 10.3.3, page 724) to just state that eventually \( 0 < \frac{1}{n} < \frac{1}{n!} \), or even \( 0 < \frac{1}{n!} < \frac{1}{2n} \), both of which are summable. However, we computed \( \rho = 0 \) which means that the terms added in the series shrink faster than any geometric series (which would have positive \( \rho \)), which is a useful insight in its own right.

Example 10.5.2 Consider the series \( \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \). We compute \( \rho \) for application of the Ratio Test again:

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|(-2)^{n+1}|}{\frac{(n+1)!}{n!}} = \lim_{n \to \infty} \frac{2^{n+1}}{2n} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{2^{n+1}}{2n} - \frac{n!}{(n+1)!} = \lim_{n \to \infty} 2^{n+1} - \frac{2}{n+1} < 1.
\]

Therefore \( \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \) converges absolutely. (Note that we also get \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \) converges.)

In fact the example above gives us an elegant proof that \( \lim_{n \to \infty} \frac{2^n}{n!} = 0 \). The example shows, by the Ratio Test, that the series \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \) converges, so (by the contrapositive of the NTTFD) the terms must shrink to zero, i.e.,

\[
\lim_{n \to \infty} \frac{2^n}{n!} = 0.
\]

The Ratio Test is clearly quite useful when there is much cancellation in \( |a_{n+1}/a_n| \), such as with factorials and some exponential functions.

Example 10.5.3 Consider the series \( \sum_{n=1}^{\infty} \frac{2^n}{n^n} \). A Ratio Test argument would be unwieldy (as the reader is invited to check), so we look instead to the Root Test:

\[
\lim_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} \left( \frac{2^n}{n^n} \right)^{1/n} = \lim_{n \to \infty} \frac{2}{n} = 0 < 1.
\]

Thus the series converges (absolutely, but that is redundant here).
As noted after the previous example, we then easily get that \(2^n / n^n \to 0\) as \(n \to \infty\), giving another of our orders in our hierarchy of functions (Theorem 10.3.3, page 724). Replacing 2 by any other number will show that \(a^n\) grows more slowly than \(n^n\). Next we see a proof that \(n^n\) grows faster than \(n!\).

**Example 10.5.4** Consider the series \(\sum_{n=1}^{\infty} \frac{n!}{n^n}\).

The \(n!\) term seems better suited to the Ratio Test, where the \(n^n\) term indicates a Root Test. Since we can more easily deal with a ratio of powers than a root of a factorial, we will opt for the Ratio Test.

\[
\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{n^n} \cdot \frac{n^n}{(n+1)!} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n.
\]

For this \(1^\infty\) form limit, we need some logarithmic arguments. We will let \(y = \left(\frac{n}{n+1}\right)^n\) and find \(\lim_{n \to \infty} \ln y\):

\[
\ln y = \ln\left(\frac{n}{n+1}\right)^n = n \ln\left(\frac{n}{n+1}\right) = \frac{\ln n - \ln(n+1)}{n^{-1}}.
\]

Using L'Hopital's Rule:

\[
\lim_{n \to \infty} \frac{\ln n - \ln(n+1)}{n^{-1}} = \lim_{n \to \infty} \frac{1/n - 1/(n+1)}{-n^{-2}} = \lim_{n \to \infty} \frac{-n^2}{n(n+1)} = -1.
\]

Thus \(\rho < 1\), and the series converges (absolutely, which is yet again redundant here). Note that we knew \(\ln n - \ln(n+1) \to 0\) because it is the same as \(\ln \frac{n}{n+1} \to \ln 1 = 0\).

We can again argue that because \(\sum \frac{n!}{n^n}\) converges, we must have

\[
\lim_{n \to \infty} \frac{n!}{n^n} = 0.
\]

We should also note that there is a Direct Comparison Test argument that this series should converge. Note that for large enough \(n\), we have

\[
0 < \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot (n)}{n \cdot n \cdot n \cdots n} = \frac{1 \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n}}{n \cdot n \cdot n \cdots n} < \frac{2\cdot 1 \cdot 1 \cdots 1 \cdot 1}{n^2} = \frac{2}{n^2}.
\]
Thus \( \frac{m}{n} \leq \frac{1}{2} \) for large \( n \), and since \( \sum \frac{a}{n}^{2} \) converges, so does \( \frac{a}{n}^{1} \).

This is a fairly common type of argument, showing that part of the fraction representing \( a_{n} \) is less than a certain size, or greater than some other size, with what is remaining representing a useful series for the DCT.

In fact, as mentioned previously, since the ratio and root tests are ultimately proved using a Direct Comparison Test (on \( \sum |a_{n}| \)), it is not surprising that there is a DCT argument which gives us the convergence result above. However it was worth considering the Ratio Test, because we are left with the knowledge that, not only is the series shrinking faster than \( \sum \frac{a}{n}^{2} \), it is in fact shrinking approximately geometrically in the infinite tail, with a ratio of \( 1/\varepsilon \approx 0.367879441 \).

**Example 10.5.5** Consider the series \( \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^{2} \cdot 4^{2} \cdot 6^{2} \cdots (2n)^{2}} \right] \).

Here we will use the Ratio Test. Note how we insert \( n + 1 \) into the formula for \( a_{n} \) to find \( a_{n+1} \), but also how, when we set it up to see what cancels, we have to look at not only the last terms multiplied in our fractions, but also the terms just before the last terms.

\[
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_{n}|} = \lim_{n \to \infty} \frac{(-1)^{n+2} \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)}{(-1)^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{2^{2} \cdot 4^{2} \cdot 6^{2} \cdots (2n)^{2}}{2^{2} \cdot 4^{2} \cdot 6^{2} \cdots (2n)^{2}}
\]

\[
\rho = \lim_{n \to \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{2} \cdot 4^{2} \cdot 6^{2} \cdots (2n)^{2}} \cdot \frac{2^{2} \cdot 4^{2} \cdot 6^{2} \cdots (2n)^{2}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}
\]

\[
\rho = \lim_{n \to \infty} \frac{2n+1}{2n+2} = \lim_{n \to \infty} \frac{2n+1}{4n^{2}+8n+4} = 0.
\]

Since \( \rho < 1 \), we have that the original series converges absolutely.

In the above, we had to notice that in the pattern \( 1 \cdot 3 \cdot 5 \cdots (2n+1) \), the terms multiplied each differ by 2, and so the term before \( (2n+1) \) would be \( (2n-1) \), and thus this product is also \( 1 \cdot 3 \cdot 5 \cdots (2n-1) \). Similarly for the \( 2^{2} \cdot 4^{2} \cdots (2n)^{2} \) term. Working backwards from the ends of patterns like these is pretty common when using the ratio test.

The next two numbered examples show how the Ratio and Root Tests are not always sufficient to determine the convergence or divergence of a series.

**Example 10.5.6** Consider the series \( \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \).

The terms in this series are somewhat similar to those in the previous example, though without alternation, which is irrelevant to the Ratio Test. So we compute \( \rho \) as before:

\[
\rho = \lim_{n \to \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}
\]

\[
\rho = \lim_{n \to \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}
\]

\[
\rho = \lim_{n \to \infty} \frac{2n+1}{2n+2} = 1.
\]
This is a case where $\rho = 1$, which is inconclusive, so we have to look elsewhere. In doing so, we will use a technique of pairing numerator and denominator factors, similar to the method in the remarks after Example 10.5.4, page 744. For this series we can write
\[
\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2n \cdot 2 \cdot 4 \cdots (2n-2)} = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2n \cdot 2 \cdot 4 \cdots 2n-2} > \frac{1}{2n} > 0,
\]
and since $\sum \frac{1}{2n}$ diverges, so does $\sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ by the DCT.

The reader is invited to try other pairings, but with the factors at hand in the numerator and denominator, it will be impossible to find a convergent, positive-term series with larger terms than our given one.

**Example 10.5.7** Let us attempt a ratio test for the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Computing $\rho$ for these two series in turn, we get
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n+1} = 1,
\]
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1.
\]

Of course the first series diverges, while the second series converges, though both have $\rho = 1$ in common. Thus, knowing $\rho = 1$ is by itself inconclusive. (Note that both the divergence of $\sum \frac{1}{n}$ and the convergence of $\sum \frac{1}{n^2}$ were ultimately proved by the Integral Test, as were the convergence or divergence of all $p$-series.)

The Ratio and Root Tests will be especially important in the next chapter, where we will need to know where series of nonconstant terms converge. These series will be of the form $\sum a_n(x - a)^n$, where $x$ is variable and $a, a_0, a_1, a_2, \cdots$ are fixed constants. Clearly such a series will converge at $x = a$, but how far from $a$ can $x$ wander and still have the series converge? (One might ask, what is the domain of a function given by $f(x) = \sum a_n(x - a)^n$?) It depends upon the terms $a_0, a_1, a_2, \cdots$. Because the $x^n$-factor in each term is geometric, it is natural to use tests which probe for comparisons to geometric series. The fact that most of the functions studied in calculus can be written in such a manner attests to the importance of such series.

It is noteworthy that we have not proven either the RAT or the ROOT. A proof of either will require careful reading, compared to earlier proofs, but we will at least give an outline of a proof of the Ratio Test (RAT).

Suppose we have a series $\sum a_n$ so that $\rho < 1$. We need to show that this implies $\sum |a_n|$ converges, i.e., $\sum a_n$ converges absolutely. Now $|a_{n+1}/a_n| \longrightarrow \rho$, so for large enough $N$, we have $n \geq N$ implies $\frac{\rho}{2} \leq |a_{n+1}/a_n| < \frac{\rho+1}{2}$, i.e., these ratios are close enough to $\rho$ to be greater than or equal to $\frac{\rho}{2}$ but no larger than the number half-way between $\rho$ and 1. If we define $S_N$ to be the $N$th partial sum of $\sum |a_n|$ (instead of the original series), then for $N \geq N$ we have the
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Inequality

\[
S_N = \sum_{n=1}^{N} |a_n|
\]

\[
= \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{N} |a_n|
\]

\[
< \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} \left[ |a_N| \left( \frac{\rho + 1}{2} \right)^{n-N} \right]
\]

We used that, for such \( n \) we have

\[
|a_{N+1}| < \frac{\rho + 1}{2} |a_N|,
\]

\[
|a_{N+2}| < \frac{\rho + 1}{2} |a_{N+1}| < \left( \frac{\rho + 1}{2} \right)^2 |a_N|,
\]

In general we get, for \( n > N' \), that

\[
|a_n| < |a_N| \left( \frac{\rho + 1}{2} \right)^{N-n},
\]

as used in the inequality for \( S_N \) above. Since the final series written in that inequality is a convergent, geometric series \( (r = (\rho + 1)/2 \in (-1, 1)) \), we have \( \{S_N\} \) is a bounded, obviously increasing sequence, and therefore converges.

One could instead note a DCT argument with the second series being \( \sum_{n=1}^{\infty} |a_N| \left( \frac{\rho + 1}{2} \right)^{N+n} \), which will have terms larger than those of the original series eventually, and still has ratio \( \frac{\rho + 1}{2} \), which implies convergence.

A similar argument can be made, somewhat modified, to show that \( \rho > 1 \implies |a_n| \neq 0 \). With \( \rho = 1 \) there is no “wiggle room” to produce the inequalities we need. Also, the argument is a bit trickier to show the same are true for the Root Test (ROOT).