## 2 Real Numbers, Algebra and Functions

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2.4 Functions-I

Behind every calculus problem looms at least one function. In fact one can credibly argue that the main goal of calculus is to provide (powerful) new tools for analyzing functions. As we will discuss momentarily, functions play a crucial role in our analysis of physical reality, and so ultimately calculus provides a (tremendous, as we will see) leap in our ability to analyze real-world problems.

However it cannot be overstated that students should also be able to spot trends in the functions without resorting to the calculus. Of particular importance is the ability to “parse” a given function, i.e., to examine the various simple components, and then “synthesize” information on how the various components would behave in isolation with how they are connected, interacting to form the final output.27

The two types of skills for analyzing functions, namely the calculus-inspired skills we begin discussing with Chapter ?? and the pre-calculus “Theory of Functions” skills discussed below in this chapter, are quite complementary and are both necessary for a complete understanding of the functions, again this being arguably the main goal of studying calculus in the first place.

Whole courses in the theory of functions used to be common, usually without any reference to the calculus. Their existence was proof that there is much intuition we can have regarding functions without resorting to calculus. We will not completely recreate a pre-calculus theory of functions here, but it is important for the student to be aware of the general principles and when they apply.

However, with calculus we have many tools which are so powerful that one is tempted to conclude calculus should obsolete the original precalculus theory of functions. This is not the case. The tools are different and complementary. In fact, once the calculus is applied we often need the pre-calculus theory of functions to interpret the calculus computation.

It should be pointed out that a discussion of functions can proceed on a very abstract level, but for most of what we do here it will be sufficient to consider functions as processes which take an input and return an output in a deterministic way, meaning that when we feed a particular function the same input on two occasions, we get the same output both times.

A function would not be terribly interesting unless it could process many different inputs. The set of inputs that a function can process is called its domain. The domain is usually apparent from the description of the function, as we will see later. The set of all possible outputs is called the range of the function, and it is usually a more difficult set to compute.

Many physical phenomena are either functions or can be very well modeled by functions. Here are some examples of functional relations:

- The area of a circle is a function of its radius.
- The voltage across an outlet is a function of time, which we will see is oscillatory.
- In a right triangle, the ratio of lengths of the side opposite one of the nonright angles to the hypotenuse is a function of the angle’s measure.

Here are some examples that are well-approximated by functions, and would be functional relations in “ideal” circumstances.

- The gain of an audio amplifier is approximately a function of the position of the volume knob. (This is true ideally, and assumes the other knobs, the qualities of the components in the circuits, the power supplied, etc., are all fixed into an unchanging position.)

27Without actively considering how a particular function is put together with its component parts—each having its own peculiar behavior—to form a single process, we can usually apply formulas but we will not be able to fully interpret their outputs, and would remove a valuable layer of anticipations and consequent error correction from our computations.
The range of an artillery projectile is approximately a function of the angle at which it is fired. (This assumes each projectile is identical, and the artillery piece, the wind, temperature, humidity, etc., are all fixed.)

- One’s net monetary worth is a function of time (though it might not be perfectly known, and opinions may vary).

Since the input can usually be varied, we usually consider it a variable. To name this variable we will usually use the obvious and descriptive term *input variable*, though most texts use the equally descriptive but more mathematically flavored term *independent variable*.

Because we can vary the input, we expect the output may also vary, and will so describe it as the *output variable*, or *value of the function*, or occasionally (here) use the more commonly found term *dependent variable*.

### 2.4.1 Various Definitions of Functions; Notation

There are many ways to describe functions. To name a couple, we can look at them as mappings (which “map” the independent variable values to their respective dependent variable outputs), and they can be described as processes or “machines.” We will include the abstract definition later, to be complete.\(^\text{28}\)

Ideally it is best to consider functions in all these ways. However for our purposes we will concentrate on the notion of functions as machines; for our purposes, a

\(^\text{28}\)Like those in other “hard” sciences, professionals and developers in the mathematical sciences over the decades and centuries have settled upon some very refined definitions of important concepts. Particularly in mathematics, a definition must be absolutely precise, because we proceed with absolute logic from our definitions. Our definitions must also be robust, for many reasons, for instance so we can recognize the same phenomena in different settings. Some find these “refined” definitions very awkward at first. Part of the process of maturing in one’s understanding of a field is the realization that, ultimately, these definitions are in fact the most “natural” for what we require of them (and we require a lot!).
function is defined by its action, by which it takes inputs from a set called its domain, and deterministically return outputs from another set called a target set, or if we know the exact set of possible outputs, we call this target set the function’s range (compare to the earlier description of range). By deterministically we mean that, for a given function, if we know its input then the output is determined; the same input cannot yield two different outputs.

As is the case for any mathematical object, it is customary to give a function a name. The most common name is \( f \), for “function,” but other letters from various alphabets are common (\( g, h, \phi, \Phi \), for some examples), and some particular functions have common names (\( \sin, \tan, \log \), etc.). See Figure 2.5

Once we have a name for the function, we then usually give names to the input and output variables. The most common name for the input variable, also known as the independent variable, is \( x \) (though \( t, \theta \), and others will be used on occasion). If \( f \) is the function, and the input variable is \( x \), then the output is usually written with the slightly unwieldy but very descriptive \( f(x) \). Indeed it helps to visualize that the input \( x \) is processed by the function \( f \), and so when we see the symbols \( f(x) \) we are looking at the final output.

Whole research papers are written regarding how students interpret \( f(x) \). Ideally the function itself is seen as an object in its own right, and then one realizes that it can be completely defined by its actions, taken collectively. In other words, if two functions always return the same output for when given the same inputs, they are considered the same functions. As is often the case, the actual mathematical definition is more abstract, and while we mention it here for further meditation, we will not use it within the rest of the text because it de-emphasizes functions as actions or processes taking inputs and deterministically returning outputs. See Footnote 28.

**Definition 2.4.1 (Abstract)** A function \( f \) is a set of ordered pairs where the first in each pair comes from a given set called the domain, say \( S \) of the function, and where

\[
(\forall x_1, x_2, y_1, y_2) \left[ (x_1, x_2 \in S) \land ((x_1, y_1), (x_2, y_2) \in f) \land (x_1 = x_2) \implies (y_1 = y_2) \right].
\]

To re-interpret in our earlier terms, the definition states that if the inputs processed by a particular function are the same \( (x_1 = x_2) \), then the outputs will be the same \( (y_1 = y_2) \). It is this deterministic nature which is key to the concept of function.

**2.4.2 Functions as Processors of Inputs**

We will not again consider the function itself as a set of ordered pairs throughout the rest of the text. However we will have much use for the concept of domain, so we will redefine it for our purposes.

**Definition 2.4.2** For a function \( f \), we define its domain to be the set of all possible inputs which \( f \) can process.

While this will be somewhat circular, we can see that the deterministic nature of a function \( f \) with domain \( S \) can be summarized as follows:

\[
(\forall x_1, x_2 \in S) [(x_1 = x_2) \implies (f(x_1) = f(x_2))].
\]

In many (but not all) cases, a function can be given by an expression, which then describes the action of the function.
Example 2.4.1 Suppose that for all \( x \in \mathbb{R} \), we define \( f(x) = x^2 + 1 \). Then
\[
\begin{align*}
f(1) &= 1^2 + 1 = 2, \\
f(2) &= 2^2 + 1 = 5, \\
f \left( \frac{1}{2} \right) &= \left( \frac{1}{2} \right)^2 + 1 = \frac{5}{4} = 1.25.
\end{align*}
\]

2.4.3 Graphs, Domain and Range

The graphical illustration of a function, when possible, is a very powerful analytical tool. When both the input and output variables are numerical, in particular real numbers, in theory a graph of the output versus the input can convey complete information about the action of the function in all circumstance. In other words, the graph can serve as a definition of the function.

Unfortunately limitations of space and resolution too often restrict our ability to plot a function with completeness and absolute precision. For that and other reasons, this textbook will not take a “graphing calculator first” approach to functions employed by many texts. However the graph of a function is a very powerful analytical tool.

There is much notation which is used in dealing with functions, even in the abstract. Some of it is useful here because of its descriptive nature. For instance, if we know that the outputs of \( f \) will always be contained in some set \( T \), we can write
\[
f : S \rightarrow T,
\]
read “\( f \) maps \( S \) into \( T \).” We can think of \( T \) as a target set for \( f \). The notation above implies that all possible inputs into \( f \) (i.e., all values in the domain \( S \)) will yield outputs in the set \( T \). There are usually many possible sets \( T \), because once we have a legitimate target set, any superset of it is also a target set. Target sets are usually easy to construct, but the actual set of outputs can sometimes be elusive because we have to consider all possible inputs. Still it is worth finding, so we can know exactly what kinds of outputs are possible. This set of all possible outputs is called the range of the function. If we denote this set by \( R \), for a particular function \( f \), then we can use our set notation to write
\[
R = \{ y \mid (\exists x \in S)[y = f(x)] \}.
\]

Another notation which is occasionally useful is the pointwise “maps to” arrow, \( \rightarrow \), where \( y = f(x) \) can also be written \( x \mapsto y \). We will have occasion to use this notation.

Note that when we write \( f : S \rightarrow T \) the arrow is not a logical implication arrow, but a visual cue that \( f \) inputs elements of \( S \) and outputs elements of \( T \).

2.4.4 Functions By Formulas

Very often the action of \( f \) on an arbitrary \( x \) in the domain will be given by a formula. For example, perhaps \( f \) acts on every real number by squaring the number and then adding 1. Then we could write
\[
f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2 + 1. \tag{2.53}
\]
This is read, “\( f \) maps \( \mathbb{R} \) into \( \mathbb{R} \), where \( f \) of \( x \) equals \( x^2 + 1 \).” The variable \( x \) in (2.53) is a placeholder, or dummy variable in which we follow through the formula to define, or even “probe” the action of \( f \) on an arbitrary input \( x \).\footnote{An elegant term found in computer science for the formula which defines a function, as in \( f(x) = x^2 + 1 \), is function prototype, in which the dummy variable \( x \) has no other significance than to describe the function’s action to the compiler.} The variable \( x \) is also often called the argument of
the function $f$ in the expression $f(x)$. Another notation which is commonly used is (notice the difference in the arrow) is
\[ x \xrightarrow{f} x^2 + 1, \] (2.54)
showing that $x$ is “transformed” or “mapped” to the value $x^2 + 1$ through the function $f$. In the notation of (2.53) we can calculate
\[
\begin{align*}
  f(1) &= (1)^2 + 1 = 2, \\
  f(2) &= (2)^2 + 1 = 5, \\
  f(-10) &= (-10)^2 + 1 = 101,
\end{align*}
\]
while in the notation of (2.54) we would write
\[
\begin{align*}
  1 \xrightarrow{f} 1^2 + 2 &= 2, \\
  2 \xrightarrow{f} 2^2 + 1 &= 5, \\
  -10 \xrightarrow{f} (-10)^2 + 1 &= 101.
\end{align*}
\]
This may seem simple enough with numbers, since we simply substituted $x = 1, 2, -10$ respectively into, say, (2.53). Our understanding of the role of $x$ in the formula becomes more important when our inputs become more complicated or abstract. For some examples, consider
\[
\begin{align*}
  f\left(\sqrt{x}\right) &= (\sqrt{x})^2 + 1 = x + 1, \\
  f\left(\frac{1}{x}\right) &= \left(\frac{1}{x}\right)^2 + 1 = \frac{1 + x^2}{x^2}, \\
  f(-x) &= (-x)^2 + 1 = x^2 + 1, \\
  f(x + 2) &= (x + 2)^2 + 1 = x^2 + 4x + 4 + 1 = x^2 + 4x + 5, \\
  f(\text{Bob}) &= (\text{Bob})^2 + 1.
\end{align*}
\]
In the above, we again replaced $x$ from (2.53) with $\sqrt{x}, \frac{1}{x}, -x, x + 2$ and Bob, respectively. This may seem unnatural until we again remember that $x$ was just a place holder in the formula which defined the action of $f$ (always one of squaring the input, and then adding 1). We also have to be careful for which values of $x$ the expression makes sense for. In the first example, because the first action on $x$ is taking its square root we require $x \geq 0$, even though the simplified expression glosses over this requirement. We must look at the original expression for $f\left(\sqrt{x}\right)$ to decide which $x$-values it is valid for. In the second example, we need $x \neq 0$, while the expressions for $f(-x)$ and $f(x + 2)$ were valid for all $x \in \mathbb{R}$. Finally, the last expression is valid as long as Bob $\in \mathbb{R}$.

We are given great latitude in defining functions. The only condition is that they are deterministic processes, so that if we know the input $x$ from the domain, then the unique output $f(x)$ is completely determined. (For each $x$ in the domain, there is exactly one $f(x)$ in the target set.) Of course the definition of $f(x)$ must also make sense for each $x$ in the domain.

### 2.5 Basic Functions

As mentioned before, it is very important to understand the basic mechanisms of the simpler functions, from which we construct more complex functions. Each function considered here will be examined from two related perspectives:
1. The actual definition, which usually contains the motive for our interest in the function;

2. The behavior of the function, by which we mean the manner in which the outputs vary with the inputs, as often summarized by the function’s graph.

In connecting these two aspects of the functions introduced below, we will be well on our way to preparing for much more complicated functions.

2.5.1 “Linear” Functions

These are functions which, when graph, yield a straight line. By the nature of functions these must be nonvertical lines, or else one input would yield more than one output (infinitely many in fact).

2.5.2 Simple Powers

We begin with the function $f(x) = x^2$. Since $f(x) = x \cdot x$, this function can process any real number and produce an output. We are interested in analyzing how this output changes when $x$ varies.

When $x = 1, 2, 3, \ldots$, it becomes clear $f(x) = x \cdot x$ will output larger and larger numbers when we input larger numbers. It is also the case when $x = -1, -2, -3, \ldots$, and a casual observation we can make is that this particular $f$ outputs the same for a positive value of $x$ as it does for the additive inverse, so for instance both $f(3)$ and $f(-3)$ are the same, namely 9. So $f(x)$ will take numbers with large absolute values and output large, positive numbers (larger than the inputs $x$ if $|x| > 1$). What is often equally important is what occurs when $x$ is a “smaller” number, for instance when $-1 < x < 1$. For instance, $f(1/2) = 1/4$, $f(1/10) = 1/100$, and so on. We note these two trends in the charts below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 1$</td>
<td>1</td>
<td>$0$</td>
<td>0</td>
</tr>
<tr>
<td>$\pm \frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\pm 2$</td>
<td>4</td>
</tr>
<tr>
<td>$\pm \frac{1}{3}$</td>
<td>$\frac{1}{9}$</td>
<td>$\pm 3$</td>
<td>9</td>
</tr>
<tr>
<td>$\pm \frac{1}{4}$</td>
<td>$\frac{1}{16}$</td>
<td>$\pm 4$</td>
<td>16</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\pm \frac{1}{10}$</td>
<td>$\frac{1}{100}$</td>
<td>$\pm 10$</td>
<td>100</td>
</tr>
<tr>
<td>$\frac{1}{100}$</td>
<td>$0.001$</td>
<td>$\pm 100$</td>
<td>10,000</td>
</tr>
<tr>
<td>$\frac{1}{1,000}$</td>
<td>$0.0001$</td>
<td>$\pm 1,000$</td>
<td>1,000,000</td>
</tr>
<tr>
<td>$\frac{1}{10,000}$</td>
<td>$0.00001$</td>
<td>$\pm 10,000$</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

While formulas and charts are useful for understanding the action of a function, another, indespensible tool is its graph, when practical. The graph of $y = x^2$ is given in Figure 2.6. Theoretically, the graph illustrates all the behavior of the function, as described above, as some contemplation of the graph illustrates. Technically, the graph’s shape is that of a parabola, though we will not define the term here.\footnote{Many shapes resemble the parabola, but there is a precise geometric definition of the term (just as there are definitions for circles, ellipses, hyperbolas, and other shapes). A common mistake is to use the term parabola (or the adjective parabolic) to apply to every “U-shaped” graph, but that is incorrect. That said, in fact any graph of a polynomial of degree two, i.e., $y = ax^2 + bx + c$ where $a \neq 0$, will be a parabola.}

We see already that it is not always easy to plot a function for a large range of the input variable. Figure 2.6 shows two partial graphs of this function. The first uses the same scale for $x$ and $y$, while the second uses very different scales for the two axes. Note that $x \in [-10, 10]$ would
Figure 2.6: Two partial graphs of the same function \( y = x^2 \) \((f(x) = x^2)\), with different scales for the axes. In the first graph, both axes use the same scale, but this is not necessarily appropriate in all circumstances. The second graph uses a different scale for \( x \) and \( y \) axes. Both shapes are parabolic. Points where \( x \) is an integer are highlighted on each graph.

require \( y \in [0, 100] \), which is not practical to graph with matched scaling in \( x \) and \( y \). Indeed, we should not always require that the input and output axes share the same scale, as they are often “incommensurable.” For instance, if \( x \) represents the length of a side of a square, then \( y = x^2 \) would represent area. If the units of \( x \) are in feet, then the units of \( y \) would need to be feet\(^2\), so indeed the units are often dissimilar. Often the horizontal axis represents a time scale and the vertical something very different, such as units of currency, power, or any other imaginable unit.

Many functions grow very rapidly, so our choices are often to plot a smaller input range, or to have a different scale for the \( y \)-axis, or perhaps to truncate the \( y \)-axis (or \( x \)-axis). The graph in Figure 2.6 actually shows the important trends in the behavior of \( f \); someone viewing the graph with the input range \([-3, 3]\) is not likely to be surprised by the behavior elsewhere, and indeed we plotted the interesting “features” of the graph. Some texts would call our partial graph “complete” because it actually presents, fairly exhaustively, all the behavior and features of the function.

We now consider other powers of \( x \). First we look at the even powers, namely \( x^2 \), \( x^4 \), \( x^6 \), etc., for comparison purposes. What we will find is that all the trends displayed by the function \( x^2 \) are also present in \( x^4 \), \( x^6 \), and so on, but they are even more pronounced. Note that all of these output the same values at \( x = \pm 1 \) and \( x = 0 \), namely 1 and 0 respectively. But when “smaller” values are input to \( x^2 \), we saw the output was smaller still, and this will also occur with the higher, even powers of \( x \). On the other hand, larger numbers will output positive even powers of larger numbers, which will be larger still, and the higher the power the more pronounced that effect will be. The following table gives some idea of these trends.
Figure 2.7: Part a shows partial graphs of $y = x^2$, $y = x^4$, $y = x^6$ and $y = x^8$. Part b shows partial graphs of $x$, $x^3$, $x^5$, $x^7$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^2$</th>
<th>$x^4$</th>
<th>$x^6$</th>
<th>$x^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm.001$</td>
<td>.000001</td>
<td>.000000000001</td>
<td>.000000000000000001</td>
<td>.000000000000000000000001</td>
</tr>
<tr>
<td>$\pm.01$</td>
<td>.001</td>
<td>.00000001</td>
<td>.000000000001</td>
<td>.000000000000000000000001</td>
</tr>
<tr>
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<td>.01</td>
<td>.001</td>
<td>.0000001</td>
<td>.000000001</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pm1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\pm2$</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
</tr>
<tr>
<td>$\pm3$</td>
<td>9</td>
<td>81</td>
<td>729</td>
<td>6561</td>
</tr>
<tr>
<td>$\pm4$</td>
<td>16</td>
<td>256</td>
<td>4096</td>
<td>65536</td>
</tr>
<tr>
<td>$\pm10$</td>
<td>100</td>
<td>10000</td>
<td>100000</td>
<td>1000000</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$\pm10$</td>
<td>10000</td>
<td>100000</td>
<td>1000000</td>
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<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

These trends are to be expected. The graphs are somewhat difficult to render because the trends occur so quickly. Figure 2.7a shows these trends, albeit incompletely.

Odd powers have similar behavior, with the exception that they are sensitive to the sign of the input. For instance, if $f(x) = x^3$, then $f(2) = 8$ while $f(-2) = -8$. Otherwise, again inputs $x \in (-1, 1)$ yield outputs of even smaller absolute values while inputs $x$ with $|x| > 1$ yield
outputs with even larger absolute values. The trends are somewhat illustrated in Figure 2.7b. We can also compare sizes of outputs of all positive integer powers, allowing for sign difference:

\[ |x| < 1 \implies |x| > |x|^2 > |x|^3 > |x|^4 > |x|^5 > \cdots, \]
\[ |x| > 1 \implies |x| < |x|^2 < |x|^3 < |x|^4 < |x|^5 < \cdots. \]

Of course, for even powers we can omit the absolute values. Also, for \( x = 0, 1 \) all powers output the same values, namely 0 and 1 respectively, and for \( x = -1 \) each power outputs \( \pm 1 \), depending upon whether the power is even or odd.

### 2.5.3 Roots

Recall that \( \sqrt[n]{x} = x^{1/n} \), for \( n = 1, 2, 3, \ldots \). Also recall that even roots input and output only nonnegative real numbers, while odd roots can input and output any real number. Now the graph of \( y = x^{1/2} \), i.e., \( y = \sqrt{x} \) is simply the upper piece of the graph \( y^2 = x \), since

\[ y = \sqrt{x} \iff (y^2 = x) \land (y \geq 0). \]

A less precise statement is \( y = \sqrt{x} \implies y^2 = x \), which shows the graph of \( y = \sqrt{x} \) is a subset of the graph of \( y^2 = x \). What makes this useful is that graphing \( y = \sqrt{x} \) is easily done in reference to \( y = x^2 \), in that \( x \) and \( y \) simply change roles, and we omit those points in the new graph in which \( y < 0 \). This is accomplished in Figure 2.5.3a, page 105.

That \( y = x^2 \) and \( y = \sqrt{x} \) are related by the role-reversals of \( x \) and \( y \), as long as \( x, y \geq 0 \), we can note some trends. On the graph \( y = x^2 \), when \( x > 1 \) (and thus \( y > 1 \)) a small upward change in the value of \( x \) can cause a much faster change in the value of \( y \). This implies that on the graph \( y = \sqrt{x} \) (as part of \( x = y^2 \)), we have large increases in \( x \) do not cause such large increases in \( y \). Indeed, \( y = \sqrt{x} \) is “grows slowly” as \( x \) grows, for \( x > 1 \). When \( x \) is near to 0 (as is \( y \)), on the other hand, \( y = x^2 \) grows very slowly as \( x \) increases, and thus \( y = \sqrt{x} \) grows very quickly as \( x \) increases from 0. Both of these trends in \( y = \sqrt{x} \), i.e., quick growth followed by slow growth, are illustrated the graph in Figure 2.5.3a.

We concentrate on the square root because it is such a common function to encounter in studying calculus. The other roots are also important, but their relative sizes are left for the exercises as straightforward extensions of our analysis of the square root. However, for completeness we include the graph of \( y = \sqrt[3]{x} \), which is exactly the same graph as \( x = y^3 \), and so is readily drawn from the graph \( y = x^3 \), with \( x \) and \( y \) roles reversed. The graph is drawn
in Figure 2.5.3b. Odd and even roots differ in that odd roots have domain \( x \in \mathbb{R} \) (and range \( y \in \mathbb{R} \)), where even roots can only input (and output) nonnegative numbers.

### 2.5.4 Reciprocal Function

The function \( f(x) = 1/x \) occurs in many contexts, alone or as a part of a more complicated function. It simply returns the reciprocal of its input. Note that this function is also an integer power, namely \( f(x) = x^{-1} \), and that it is undefined at \( x = 0 \).

The most interesting features of this function include what occurs when \( x \) is near zero, and what occurs for \(|x|\) large. First, reciprocals of “small numbers,” such as \( \pm 0.1, \pm 0.01, \pm 0.001 \) and so on, are in fact larger numbers \( \pm 10, \pm 100, \pm 1000 \), respectively, and so on. On the other hand, reciprocals of “large numbers,” such as \( \pm 10, \pm 100, \pm 1000 \) are small numbers \( \pm 0.1, \pm 0.01, \pm 0.001 \). The reciprocal function preserves the sign, meaning a positive input yields a positive output, while a negative input yields a negative output. Because the output’s growth near zero occurs quickly, a graph with \( x \) and \( y \) using the same scale is difficult to produce showing these trends with a lot of precision. In Figure 2.9a, page 107, the trends are shown but with more easily seen inputs and outputs.

The way the output of \( y = 1/x \) “blows up” as \( x \) nears zero is geometrically described as causing the graph to have a vertical asymptote at \( x = 0 \). On the other hand, the way that the output nears 0 as \( x \) becomes a large positive number, or a large negative number, is geometrically described as causing the graph to have a two-sided horizontal asymptote \( y = 0 \) as \( |x| \) grows large. By two-sided we mean there is a horizontal asymptote for \( x \) growing larger and positive, and the same line for an asymptote for \( x \) growing larger and negative. Asymptotes and asymptotic behavior of functions is very important, and will be described in more detail later in the text. For now, we simply mention that an asymptote is a shape which the graph grows closer and closer, without necessarily touching, for some movement in \( x \). For our present example, the graph becomes more “vertical” and increasingly close in shape and behavior to the vertical line \( x = 0 \) for small \( x \), and becomes more “horizontal” and increasingly close to the horizontal line \( y = 0 \) for large \( x \).

### 2.5.5 Absolute Value and Piece-wise Defined Functions

The absolute value is naturally the first of the piecewise-defined functions one usually studies. It is defined by

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0.
\end{cases}
\]

The graph is given in Figure 2.9a.

### 2.5.6 Sine and Cosine Functions

All trigonometric functions trace their ancestry to the sine and cosine functions. And these functions can be characterized simply as describing the variations in, respectively, vertical and horizontal positions of a moving point on the unit circle, \( x^2 + y^2 = 1 \), as the point travels with uniform speed along the circle counter-clockwise. The position of the point is determined by the angle \( \theta \), measuring the angle traveled from the starting point on the positive \( x \)-axis, i.e., from \((1, 0)\). This is shown in Figure 2.10, with the angles measured in both degrees and radians.

While they may not seem intuitive at first, in fact radians are more “natural” than the artificial measure of degrees, where convention put a whole rotation at 360° with little explanation. “Radians” are actually measures of arc-length where the units are radius lengths.
2.5. BASIC FUNCTIONS

Figure 2.9: Partial graphs of \( f(x) = \frac{1}{x} \) and \( f(x) = |x| \). Figure 2.9a shows \( y = \frac{1}{x} \), which maps large numbers to small, and vice versa, while preserving the sign. The graph is said to show a vertical asymptote which is the line \( x = 0 \), and a two-sided horizontal asymptote which is \( y = 0 \). Figure 2.9b shows how \( y = |x| \) can be defined piecewise to be \( y = -x \) for \( x < 0 \) and \( y = x \) for \( x \geq 0 \). Its domain is \( x \in \mathbb{R} \) and range is \( y \geq 0 \).
(cos θ, sin θ)

(r cos θ, r sin θ)

x^2 + y^2 = 1

θ = s/r

s = rθ

x^2 + y^2 = r^2

θ in radians

Figure 2.10: The unit circle, $x^2 + y^2 = 1^2$, showing the point of intersection with the terminal ray of θ being $(x, y) = (1 \cos \theta, 1 \sin \theta) = (\cos \theta, \sin \theta)$. Also shown are graphs of sin θ and cos θ, where θ is measured in degrees.
2.5.7 Arcsine and Arccosine Functions
2.5.8 Exponential Functions
2.5.9 Logarithmic Functions
1. Sketch a rough graph of \( y = \sqrt[3]{x} \) based upon the graph of \( y = x^3 \). Note that the domain is \( x \in \mathbb{R} \).

2. Using one pair of axes, sketch graphs of \( y = \sqrt{x} \), \( y = \sqrt[3]{x} \), and \( y = \sqrt[5]{x} \), displaying their shapes and relative positions for relevant ranges of \( x \).

3. Using one pair of axes, sketch graphs of \( y = \sqrt{x} \), \( y = \sqrt[3]{x} \), and \( y = \sqrt[5]{x} \), displaying their shapes and relative positions for relevant ranges of \( x \).

2.6 Functions-III: Combinations of Functions