Chapter 7

Advanced Integration Techniques

Before introducing the more advanced techniques, we will look at a shortcut for the easier of the substitution-type integrals. Advanced integration techniques then follow: integration by parts, trigonometric integrals, trigonometric substitution, and partial fraction decompositions.

7.1 Substitution-Type Integration by Inspection

In this section we will consider integrals which we would have done earlier by substitution, but which are simple enough that we can guess the approximate form of the antiderivatives, and then insert any factors needed to correct for discrepancies detected by (mentally) computing the derivative of the approximate form and comparing it to the original integrand. Some general forms will be mentioned as formulas, but the idea is to be able to compute integrals without resorting to writing the usual $u$-substitution steps.

Example 7.1.1 Compute $\int \cos 5x \, dx$.

Solution: We can anticipate that the approximate form\footnote{In this section, by approximate form we mean a form which is correct except for multiplicative constants.} of the answer is $\sin 5x$, but then

$$\frac{d}{dx} \sin 5x = \cos 5x \cdot \frac{d}{dx} (5x) = \cos 5x \cdot 5 = 5 \cos 5x.$$  

Since we are looking for a function whose derivative is $\cos 5x$, and we found one whose derivative is $5 \cos 5x$, we see that our candidate antiderivative $\sin 5x$ gives a derivative with an extra factor of 5, compared with the desired outcome. Our candidate antiderivative’s derivative is 5 times too large, so this candidate $\sin 5x$ must be 5 times too large. To compensate and arrive at a function with the proper derivative, we multiply our candidate $\sin 5x$ by $\frac{1}{5}$. This gives us a new candidate antiderivative $\frac{1}{5} \sin 5x$, whose derivative is of course $\frac{1}{5} \cos 5x \cdot 5 = \cos 5x$, as desired. Thus we have

$$\int \cos 5x \, dx = \frac{1}{5} \sin 5x + C.$$  

It may seem that we wrote more in the example above than with the usual $u$-substitution method, but what we wrote could be performed mentally without resorting to writing the details.

In future sections, an integral such as the above may occur as a relatively small step in the execution of a more advanced and more complicated method (perhaps for computing a much
more difficult integral). This section’s purpose is to point out how such an integral can be quickly dispatched, to avoid it becoming a needless distraction in the more advanced methods.

Some formulas which should be quickly verifiable by inspection (that is, by reading and mental computation rather than with paper and pencil, for instance) follow:

\[
\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C, \quad (7.1)
\]
\[
\int \cos kx \, dx = \frac{1}{k} \sin kx + C, \quad (7.2)
\]
\[
\int \sin kx \, dx = -\frac{1}{k} \cos kx + C, \quad (7.3)
\]
\[
\int \sec^2 kx \, dx = \frac{1}{k} \tan kx + C, \quad (7.4)
\]
\[
\int \csc^2 kx \, dx = -\frac{1}{k} \cot kx + C, \quad (7.5)
\]
\[
\int \sec kx \tan kx \, dx = \frac{1}{k} \sec kx + C, \quad (7.6)
\]
\[
\int \csc kx \cot kx \, dx = -\frac{1}{k} \csc kx + C, \quad (7.7)
\]
\[
\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + C. \quad (7.8)
\]

Example 7.1.2 The following integrals can be computed with u-substitution, but also are computable by inspection:

\[
\int \frac{1}{5x-9} \, dx = \frac{1}{5} \ln |5x-9| + C,
\]
\[
\int \sin 5x \, dx = -\frac{1}{5} \cos 5x + C,
\]
\[
\int \cos \frac{x}{2} \, dx = 2 \sin \frac{x}{2} + C,
\]
\[
\int \sec^2 \pi x \, dx = \frac{1}{\pi} \tan \pi x + C,
\]
\[
\int \csc 6x \cot 6x \, dx = -\frac{1}{6} \csc 6x + C.
\]

While it is true that we can call upon the formulas (7.1)–(7.8), the more flexible strategy is to anticipate the form of the antiderivative and adjust accordingly. For instance, we have the following antiderivative form, written two ways:

\[
\int \frac{1}{u} \, du = \ln |u| + C,
\]
\[
\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C.
\]

(As usual, the second form is the same as the first where \( u = f(x) \).) So when we see an integrand which is a fraction, the numerator being the derivative of the denominator except for multiplicative constants, we know the antiderivative will be, approximately, the natural log of the absolute value of that denominator.
Example 7.1.3 Consider \( \int \frac{x}{x^2 + 1} \, dx \)

Here we see that the derivative of the denominator is also a factor in the integrand. Our candidate approximate form can then be \( \ln |x^2 + 1| = \ln(x^2 + 1) \). Now we differentiate to see what we need to include to get the correct derivative:

\[
\frac{d}{dx} \ln(x^2 + 1) = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}.
\]

To correct for the extra factor of 2 and thus get the correct derivative, we insert the factor \( \frac{1}{2} \):

\[
\frac{d}{dx} \left[ \frac{1}{2} \ln(x^2 + 1) \right] = \frac{1}{2} \cdot \frac{1}{x^2 + 1} \cdot 2x = \frac{x}{x^2 + 1},
\]

as desired. Thus

\[
\int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \ln(x^2 + 1) + C.
\]

To be sure, a quick (mental?) check by differentiation verifies the answer.

Of course there are many other forms.

Example 7.1.4 Consider \( \int \frac{1}{\sqrt{5x - 9}} \, dx \).

Of course this can be rewritten \( \int (5x - 9)^{-1/2} \, dx \). Now it is crucial that a complete substitution, \( u = 5x - 9 \Rightarrow du = 5 \, dx \), etc., shows that \( du \) and \( dx \) agree except for a multiplicative constant, so we know that the integral—up to multiplicative constants—is of form \( \int u^{-1/2} \, du \), which is a power rule.

The approximate form of the antiderivative is thus \( u^{1/2} = (5x - 9)^{1/2} \), which we write in \( x \) and then differentiate,

\[
\frac{d}{dx} (5x - 9)^{1/2} = \frac{1}{2} (5x - 9)^{-1/2} \cdot 5,
\]

which has extra factors (compared to our original integrand) of collectively \( \frac{5}{2} \). To cancel their effects we include a factor \( \frac{2}{5} \) in our actual, reported antiderivative. Thus

\[
\int \frac{1}{\sqrt{5x - 9}} \, dx = \frac{2}{5} (5x - 9)^{1/2} + C = \frac{2}{5} \sqrt{5x - 9} + C.
\]

Note that a quick derivative computation, albeit involving a (simple) chain rule, gives us the correct function \( 1/\sqrt{5x - 9} \).

Example 7.1.5 Consider \( \int 7x \sin^5 x^2 \cos x^2 \, dx \).

For such an antiderivative, our ability to guess the form depends upon our experties with the original substitution methods. In all of these was a form \( \int f(u) \, K \, du \), where we could anticipate both \( u \) and \( f \), with \( du \) accounting for remaining terms, and \( K \in \mathbb{R} \) which we can ignore by taking our shortcut path described in this section. Looking ahead, the student well-versed in substitution will expect \( u = \sin x^2 \), and the integral being of the approximate form \( \int u^5 \, du \). Thus we will have an approximate antiderivative of \( u^6 \) (times a constant), i.e., the approximate form should be \( \sin^6 x^2 \). Now we differentiate this and see what compensating factors must be included to reconcile with the original integrand:

\[
\frac{d}{dx} (\sin x^2)^6 = 6(\sin x^2)^5 \cdot \cos x^2 \cdot 2x = 12x \sin^5 x^2 \cos x^2.
\]
Of course we want \( 7 \) in the place of the \( 12 \) (or separately, \( 2 \cdot 6 \)), so we multiply by \( \frac{7}{12} \) (or again, \( 7 \cdot \frac{1}{\sqrt{2}} \)). With this we have

\[
\int 7x \sin^5 x^2 \cos x^2 \, dx = \frac{7}{12} \sin^6 x^2 + C.
\]

It would be perfectly natural to forego this method of “guess and adjust” in favor of the old-fashioned substitution method. Indeed the full substitution method has some advantages (see the next subsection). For instance, it is more “constructive,” and thus less error-prone; one is less tempted to skip steps while employing substitution, while one might attempt a mental derivation of the answer here and thus easily be off by a factor. It is important that each student find the comfortable level of brevity for himself.\(^3\)

The method used in the above examples can be summarized as follows:

1. Anticipate the form of the antiderivative by an approximate form (correct up to a multiplicative constant).

2. Differentiate this approximate form and compare to the original function (to be integrated);

3. If Step 1 is correct, and thus the approximate form’s derivative differs from the original (integrand) function by a multiplicative constant, insert a compensating, reciprocal multiplicative constant in the approximate form to arrive at the actual antiderivative;

4. For verification, differentiate the answer to see if the original function emerges.

Example 7.1.6 Compute \( \int x^3 \sin x^4 \, dx \).

Solution: This is of the approximate form \( \int \sin u \, du \), with \( u = x^4 \). The approximate form of the solution is thus \( \cos x^4 + C \) (or \( -\cos x^4 + C \), but these differ by a multiplicative constant \(-1\)), which has derivative \(-\sin x^4 \cdot 4x^3 \). We introduce a factor of \(-\frac{1}{4}\) to compensate for the extra factor of \(-4\):

\[
\int x^3 \sin x^4 \, dx = -\frac{1}{4} \cos x^4 + C,
\]

which can be quickly verified by differentiation.

Example 7.1.7 Compute \( \int x \sqrt{9 - x^2} \, dx \).

Solution: It is advantageous to read this integral \( \int x(9 - x^2)^{1/2} \, dx \), which is of approximate form \( \int u^{1/2} \, du \) (where \( u = 9 - x^2 \)). These observations, and the approximate form \((9 - x^2)^{3/2}\) of the integral, can be gotten by mental observation (referred to earlier as “by inspection”). Its derivative is \( \frac{3}{2}(9 - x^2)^{1/2} \cdot (-2x) \), which has an extra factor of \(-3\) (after cancellation). Thus

\[
\int x \sqrt{9 - x^2} = -\frac{1}{3} (9 - x^2)^{3/2} + C.
\]

\(^2\)Notice that we are assuming fluency in the chain rule as we compute the derivative of \( \sin^6 x^2 \), rather than writing out every step as we did in Chapter 4. Each student must gauge personal ability to omit steps.

\(^3\)It is the author’s experience that students in engineering and physics programs are more interested in arriving at the answer quickly, while mathematics and other science students prefer the presentation of the full substitution method. The latter are somewhat less likely to be wrong by a multiplicative constant, though the former tend to progress through the topics faster. There are, of course, spectacular exceptions, and each group benefits from camaraderie with the other.
7.1.1 Limitations of the Method

There are two very important points to be made about the limitations of the method. The first point is argued by making several related points, and the second is illustrated in an example.

(I) **This method can not totally replace the earlier substitution method.**

(a) The skills used in the substitution method will be needed for later methods. In particular, the idea that the entire integral in \( x \) is replaced by one in \( u \) (for instance), including the \( dx \) and, if a definite integral, the interval of integration.

(b) If an integral is difficult enough, the more constructive substitution method is less error-prone than this “guess and adjust” style here.

(c) The idea of the substitution method is the same as this method; anticipating what to set equal to \( u \) is equivalent to guessing the approximate form of the integral in \( u \), and thus the approximate form of the antiderivative.

(d) When using numerical and other methods with definite integrals, a substitution can sometimes make for a much simpler integral to be approximated or otherwise analyzed, even if the antiderivative is never computed. For instance, with \( u = x^2 \), giving then \( du = 2x \, dx \), we can write

\[
\int_{-1}^{2} x e^{x^4} \, dx = \frac{1}{2} \int_{1}^{4} e^{u^2} \, du.
\]

(II) **It is imperative that the derivative of the approximate form differs from the original function to be integrated by at most a multiplicative constant.** In particular, an extra variable function cannot be compensated for.

To illustrate this point, and simultaneously warn against a common mistake, consider

\[
\int \frac{1}{x^2 + 1} \, dx.
\]

The mistake to avoid here is to take the approximate solution to be \( \ln(x^2 + 1) \), which we then notice has derivative

\[
\frac{d}{dx} \ln(x^2 + 1) = \frac{1}{x^2 + 1} \cdot 2x.
\]

Unfortunately we cannot compensate by dividing by the extra factor \( 2x \), because\(^4\)

\[
\frac{d}{dx} \left[ \frac{\ln(x^2 + 1)}{2x} \right] = 2x \cdot \frac{d\ln(x^2 + 1)}{dx} - \ln(x^2 + 1) \cdot \frac{d(2x)}{(2x)^2},
\]

which is guaranteed (by the presence of the logarithm in the result) to be something other than our original function \( \frac{1}{x^2 + 1} \). The method does not work because multiplicative functions do not “go along for the ride” in derivative (or antiderivative) problems the way multiplicative constants do.

\(^4\)Alternatively, a product rule computation can be used:

\[
\frac{d}{dx} \left[ \frac{1}{2x} \ln(x^2 + 1) \right] = \frac{1}{2x} \cdot \frac{d\ln(x^2 + 1)}{dx} + \ln(x^2 + 1) \cdot \frac{1}{2x},
\]

which eventually gives the original function for the first product, but the second part of the product rule is a complication we cannot rid ourselves of easily.

It should be pointed out that the method of the next section does utilize the fact that the first product above is the desired original function, and an algorithm can be fashioned to compensate for the presence of the second product. The application of that method is not universally useful, and even when it is helpful it takes considerable work to develop the theory as well as fluency in its application.
Of course we knew from before that
\[ \int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C, \]
so this integral is not really suitable for a substitution argument, but is rather a special case in and of itself.

**Exercises**

1. \[ \int (x^2 + 1)^7 \cdot 2x \, dx \]
2. \[ \int \cos x^4 \cdot 4x^3 \, dx \]
3. \[ \int 15x^2 \sec^2 5x^3 \, dx \]
4. \[ \int \frac{\sec \sqrt{x} \tan \sqrt{x}}{2\sqrt{x}} \, dx \]
5. \[ \int \csc^2 \left( \frac{1}{x} \right) \, dx \]
6. \[ \int \tan^7 x \sec^2 x \, dx \]
7. \[ \int \frac{x}{(x^2 + 1)^3} \, dx \]
8. \[ \int (2x - 11)^9 \, dx \]
9. \[ \int \cos 5x \, dx \]
10. \[ \int \sec 9x \tan 9x \, dx \]
11. \[ \int \cos x \sin x \, dx \] (See #13)
12. \[ \int \tan^3 5x \sec^2 5x \, dx \]
13. \[ \int \sin x \cos x \, dx \] (See #11)
14. \[ \int \sin^3 x \cos x \, dx \]
15. \[ \int \tan^5 x \sec^2 x \, dx \]
16. \[ \int x \sin x^2 \, dx \]
17. \[ \int x^3 \cdot \sqrt{x^4 - 2} \, dx \]
18. \[ \int (x^3 + x^2)^4 (3x^2 + 2x) \, dx \]
19. \[ \int \sec^5 x \cdot \sec x \tan x \, dx \]
20. \[ \int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx \]
21. \[ \int x^2 \sin x^3 \cos x^3 \, dx \]
22. \[ \int \sin^3 \left( \frac{1}{x} \right) \cos \left( \frac{1}{x} \right) \, dx \]
23. \[ \int e^x \cos e^x \, dx \]
24. \[ \int xe^{x^2} \, dx \]
25. \[ \int e^{2x} \sin e^{2x} \, dx \]
26. \[ \int e^{-x} \sec^2 e^{-x} \, dx \]
27. \[ \int e^{5x} \, dx \]
28. \[ \int \frac{e^x}{e^{2x} + 1} \, dx \]
29. \[ \int \frac{e^x}{\sqrt{e^{2x} - 1}} \, dx \]
30. \[ \int \frac{dx}{\sqrt{e^{2x} - 1}} \] (Hint: multiply integrand by \( e^x / e^x \).)
31. \[ \int e^{4x} (9 + e^{4x})^{10} \, dx \]
32. \[ \int xe^{-2x^2} \, dx \]
33. \[ \int \frac{e^{1/x}}{x^2} \, dx \]

34. \[ \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \]

35. \[ \int e^{3 \cos 2x} \sin 2x \, dx \]

36. \[ \int \frac{\cos x}{\sin x + 1} \, dx \]

37. \[ \int \frac{\cos x}{\sin x} \, dx \]

38. \[ \int \frac{\sin x}{\cos x} \, dx \]

39. \[ \int \frac{2x + 1}{x^2 + x} \, dx \]

40. \[ \int \frac{x}{x^2 + 1} \, dx \]

41. \[ \int \frac{1}{x^2 + 1} \, dx \]

42. \[ \int \frac{1}{x \ln x} \, dx \]

43. \[ \int \frac{e^{2x}}{1 + e^{2x}} \, dx \]

44. \[ \int \frac{e^{2x}}{1 + e^{2x}} \, dx \]

45. \[ \int \frac{\sec^2 x}{1 + \tan x} \, dx \]

46. \[ \int \frac{\sin(\ln x)}{x} \, dx \]

47. \[ \int \frac{\ln x}{x} \, dx \]

48. \[ \int \frac{1}{x \sqrt{1 - (\ln x)^2}} \, dx \]

49. \[ \int \frac{1}{x(1 + (\ln x)^2)} \, dx \]

50. \[ \int \frac{1}{x|\ln x|\sqrt{(\ln x)^2 - 1}} \, dx \]

51. \[ \int \frac{\sec^2(\ln x)}{x} \, dx \]

52. \[ \int \frac{(9 + \ln x)^6}{x} \, dx \]

53. \[ \int \frac{1}{x(\ln x)^2} \, dx \]