Representations of the Euclidean Group

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Overview

1. Introduction

2. Concrete Example

3. Induced Representation

4. Big Theorem
Suppose $G$ is a group and we want to understand its representations.

Suppose further that $H \leq G$ and we already know the representations for $H$.

Is there any way to use these representations of $H$ to construct a representation of $G$?
Let $G$ be any metrizable locally compact group. We will say $G$ is an LC group for short.

$(\pi, V_\pi)$ is a unitary representation of $G$ if all of the following conditions hold.

1. $V_\pi$ is a Hilbert space.
2. $\pi : G \to GL(V_\pi)$ is a homomorphism.
3. The map $G \times V_\pi \to V_\pi$ with $(g, v) \mapsto \pi(g)v$ is continuous.
4. $\langle \pi(g)u, \pi(g)v \rangle = \langle u, v \rangle$ holds for all $g \in G$ and $u, v \in V_\pi$. 
We say a closed subspace $W \subset V_\pi$ is invariant if

$$\pi(g)w \in W$$

for all $g \in G$ and all $w \in W$.

$\pi$ is called an irreducible representation if $V_\pi$ has no proper, non-trivial invariant subspaces.
We say two representations \((\pi, V_\pi)\) and \((\rho, V_\rho)\) are *equivalent* if there is a unitary operator \(A : V_\pi \rightarrow V_\rho\) such that

\[
\begin{array}{ccc}
V_\pi & \xrightarrow{A} & V_\rho \\
\pi(g) & \downarrow & \rho(g) \\
V_\pi & \xrightarrow{A} & V_\rho
\end{array}
\]

commutes for every \(g \in G\).
We let $\hat{G}$ denote the set of equivalence classes of unitary irreducible representations of a group $G$.

Some classes of groups where $\hat{G}$ has been studied extensively:

- **finite groups** – every representation acts in a finite dimensional space
- **compact groups** – $\hat{G}$ is discrete
- **locally compact abelian groups** – irreducible representations are one-dimensional.
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We consider the full group of isometries on $\mathbb{R}^2$ containing rotations and translations.

These operations commute amongst themselves, but not with each other.

The representations of $T$ and $\mathbb{R}^2$ are already well understood.
Notation for this Euclidean Group

We take the natural action of $T$ on $\mathbb{R}^2$ given by

$$R(\theta)x = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{bmatrix}.$$ 

The action of $\mathbb{R}^2$ on itself will be translation.
We take elements of the full Euclidean group to have the form

\[(R(\theta), v)\]

where \(R(\theta)\) is a rotation matrix and \(v \in \mathbb{R}^2\).

If \(x \in \mathbb{R}^2\), we define the action

\[(R(\theta), v) \cdot x = R(\theta)x + v.\]
This is a non-Abelian Group

We calculate that

\[\left[(R(\phi), w)(R(\theta), v)\right] \cdot x = (R(\phi), w) \cdot [R(\theta)x + v] = R(\phi + \theta)x + R(\phi)v + w.\]

The other order gives

\[\left[(R(\theta), v)(R(\phi), w)\right] \cdot x = (R(\theta), v) \cdot [R(\phi)x + w] = R(\phi + \theta)x + R(\theta)w + v.\]
Conjugation in $G = \mathbb{R}^2 \rtimes T$

We note that $(R(\theta), v)^{-1} = (R(-\theta), -R(-\theta)v)$.

Conjugation in our group is non-trivial.

However, if we conjugate and element of $\mathbb{R}^2 \leq G$, things simplify considerably. We compute that

$$(R(\theta), v)^{-1}(1, w)(R(\theta), v) = (1, R(-\theta)w).$$

Thus, we see that $\mathbb{R}^2$ is a normal subgroup of $G$. 

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The key elements of creating representations for $G = N \rtimes H$:

1. Use the $G$ action on $N$ to create a $G$ action on $\hat{N}$.
2. Consider the $G$-orbits in $\hat{N}$.
3. Account for any stabilizers.
4. Form the induced representations.
5. Inducing from elements in the same orbit will yield equivalent representations.
Induced Representations of $G = \mathbb{R}^2 \rtimes T$

**Fact:** $\hat{\mathbb{R}}^2 \cong \mathbb{R}^2$ and each representation has the form

$$\chi_y(x) = e^{2\pi i \langle y, x \rangle}$$

for some $y \in \mathbb{R}^2$.

We define a $G$ action on $\hat{\mathbb{R}}^2$ by

$$(R(\theta), v) \cdot \chi_y(x) = (R(\theta), v) \cdot \chi_y(1, x)$$

$$= \chi_y \left[ (R(\theta), v)^{-1} (1, x) (R(\theta), v) \right]$$

$$= \chi_y (1, R(-\theta)x)$$
We can now see that \((R(\theta), \nu) \cdot \chi_y(x) = \chi_y(R(-\theta)x)\).

Using the definition of \(\chi\) and the fact that \(R(-\theta)\) is an orthogonal matrix, we see that

\[
(R(\theta), \nu) \cdot \chi_y(x) = \chi_y(R(-\theta)x) \\
= e^{2\pi i \langle y, R(-\theta)x \rangle} \\
= e^{2\pi i \langle R(\theta)y, x \rangle} \\
= \chi_{R(\theta)y}(x)
\]
By the above, we can see that there are two classes of $G$ orbits.

1. Circles, radius $r$, centered at the origin
2. The singleton $\{0\}$.

We make the distinction because each class has a different stabilizer.
The Circular Orbits with $r > 0$

Given any $\chi_y \in \hat{N} \setminus \{0\}$, the stabilizer is $\mathbb{R}^2 \leq G$.
The only element of $T$ that fixes $\chi_y$ is the identity.
Thus, we get our irreducible unitary representations of $G$ by

$$\pi_y = \text{ind}_{\mathbb{R}^2}^G (\chi_y)$$

and $\pi_y \cong \pi_b$ if and only if $\|y\| = \|b\|$.
The Little Group

Suppose that we are creating representations of $G = N \rtimes H$. In general, if $\chi \in \hat{N}$, we let $G_\chi$ denote the stabilizer.

The *little group* $H_\chi$ is defined to be

$$H_\chi = H \cap G_\chi$$

and we form or representations for $G$ by defining

$$\pi = \text{ind}_G^{G_\chi} (\chi \rho)$$

where $\rho \in \hat{H}_\chi$. 
Here the stabilizer is all of $G$. Thus, the little group is all of $T$.

**Fact:** $\hat{T} \cong \mathbb{Z}$ with the representations $g_n$ given by $t \mapsto t^n$.

The representations of $G$ from this orbit are given by

$$\pi_n = g_n$$

Thus, the representations of $G$ are given by this family:

$$\{\pi_r | r > 0\} \cup \{\pi_n | n \in \mathbb{Z}\}$$

According to the following theorem, these exhaust $\hat{G}$. 
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Theorem Concerning Induced Representation

Suppose $G = N \rtimes H$ where $N$ is abelian and $G$ acts regularly on $\hat{N}$. If $\nu \in \hat{N}$ and $\rho$ is an irreducible representation of $H_\nu$, then $\text{ind}_{G_\nu}^G (\nu \rho)$ is an irreducible representation of $G$, and every irreducible representation of $G$ is equivalent to one of this form. Further, $\text{ind}_{G_\nu}^G (\nu \rho)$ and $\text{ind}_{G_{\nu'}}^G (\nu' \rho')$ are equivalent if and only if $\nu$ and $\nu'$ belong to the same orbit, say $\nu' = x\nu$ and $h \mapsto \rho(h)$ and $h \mapsto \rho'(x^{-1}hx)$ are equivalent representations of $H_\nu$. 